A TALE OF TWO BEAMS: GAUSSIAN BEAMS AND BESSEL BEAMS

ROBERT L. NOWACK*

Abstract. An overview is given of two types of focused beams, Gaussian beams and Bessel beams. First I describe some of the basic properties of Gaussian beams in homogeneous media which stay collimated over a certain distance range after which they diverge. Bessel beams are then described which are among a class of solutions to the wave equation that are diffraction-free and do not diverge when they propagate. For pulsed signals, another solution that propagates in an undistorted fashion is called an X-wave. I will then compare and contrast these different beam solutions to the wave equation.

1. Introduction. Gaussian beams are focused wave solutions to the wave equation that stay collimated out to some distance range after which they diverge. In the 1980's, Durnin (1987) and Durnin et al. (1987) showed that realistic beams could propagate without change of shape to a large range in comparison to Gaussian beams and were called Bessel beams. There solutions were found earlier by Stratton (1941, pp. 356), Courant and Hilbert (1966, Vol. 2, pp. 760, Bateman (1915) among others, and an overview is given in Recami et al. (2008). However, these solutions are endowed with infinite energy, similar to plane waves, and did not attract much interest at the time.

In this overview, I will first describe some of the basic properties of Gaussian beams as examples of beams that diffract and then describe Bessel beams as examples of beam modes that are diffraction free.

2. Gaussian Beams in Homogeneous Media. Gaussian beams can be derived in several ways (Siegman, 1986). These include the complex source point approach in which an analytic continuation of a point source from a real source location x_3^0 to $x_3^0 - \mathrm{i}b$ is performed. The solution for a point source $\mathrm{e}^{\mathrm{i}kR}/R$ is then modified to a Gaussian beam with $R = [(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + (x_3 - x_3^0)^2]^{1/2}$ with a complex x_3^0 (Deschamps, 1971; Felsen, 1976). This approach can be used to extend analytical results for point sources to Gaussian beams. Other approaches to derive Gaussian beams are the differential equation approach based on the "paraxial" wave equation, the Huygens-Fresnel integral with an initial Gaussian amplitude profile, a plane wave expansion approach, and solutions to the Helmholtz equation in oblate spheroidal coordinate systems.

The field of a Gaussian beam can be written as (Siegman, 1986)

$$u(\rho, x_3) = (2/\pi)^{1/2} \frac{\mathrm{e}^{\mathrm{i}k\,x_3}}{W(x_3)} \mathrm{e}^{-\psi(x_3)} \mathrm{e}^{\frac{-\rho^2}{W^2(x_3)}} \mathrm{e}^{\frac{\mathrm{i}k\rho^2}{2R(x_3)}} \tag{2.1}$$

where $k = \omega/v$ is the wavenumber, $\rho^2 = x_1^2 + x_2^2$, W_0 is the initial beam width where the amplitude decays to 1/e in lateral distance ρ. For larger x_3 distances, $W(x_3) = W_0(1 + (\frac{x_3}{X_3^R})^2)^{1/2}$, where $X_3^R = \pi W_0^2/\lambda$ is called the Rayleigh distance. At this range the amplitude decays to $1/e$ at $\rho = \sqrt{2}W_0$. The radius of curvature of the beam is $R(x_3) = x_3\left(1 + \left(\frac{X_3^R}{x_3}\right)^2\right)$. The distance $x_3 = 0$ is called the beam waist where the beam is narrowest and the phase front is planar with $R(x_3) = \infty$. As $x_3 \to \infty$, the radius of curvature also goes to infinity. The radius of curvature is smallest (maximum curvature) at the Rayleigh distance $R(x_3)$. $\psi(x_3)$ is called the Gouy phase, and Gouy showed in 1890 that all waves going through a focus experience a π phase advance. For a Gaussian beam, $\psi(x_3) = \tan^{-1}(\frac{x_3}{x_3^R})$.

Figure 1 summaries the characteristics of a Gaussian beam in a homogeneous media. θ in Figure 1 is the far field spread of the beam where $\theta = \frac{\lambda}{\pi W_0}$. The collimated part of the beam is between $-x_3^R < x_3 < x_3^R$ where x_3^R is the Rayleigh distance. Figure 2 shows that as the initial beam width W_0 gets smaller, the Rayleigh distance also gets smaller and the far field angular spread of the beam θ gets larger.

*Dept. of Earth and Atmos. Sci., Purdue University, West Lafayette, IN 47907, USA (nowack@purdue.edu).

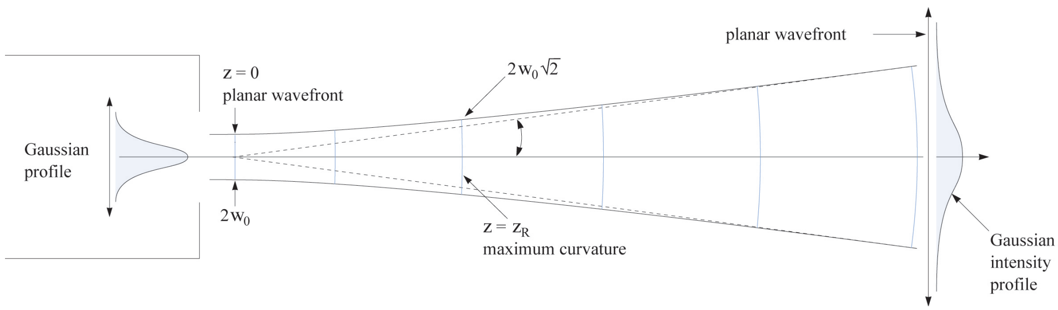

FIG. 1. *A summary of Gaussian beam propagation (from the www.mellesgriot.com website on Gaussian beam optics).*

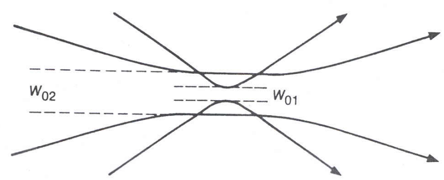

FIG. 2. *Diffraction spreading of two Gaussian beams with different spot sizes at the beam waist (from, Siegman, 1986, Figure 17.6).*

The oblate speroidal coordinate system well represents the shape of a Gaussian beam and can be used to derive Gaussian solutions to the Helmholtz equation. There coordinates were originally used for antenna theory, e.g. Stratton (1956) and Flammer (1957), and a recent overview of wave solutions in oblate spheroidal coordinates is given by McDonald (2002).

All waves that go through a focus experience a phase advance called the Guoy phase $\psi(x_3)$. Feng and Winful (2001) inferred that for a Gaussian beam, this results from the lateral beam spread as the wave eminates from the beam waist. Instead of the relation $k_3 = \omega/v = k$, the average x_3 component of the wavenumber across the beam is, $\overline{k_3} \approx k - \frac{\overline{k_1^2}}{k} - \frac{\overline{k_2^2}}{k}$, where $\overline{k_1^2}$ and $\overline{k_2^2}$ are averages of the lateral squared wavenumber components across the beam. Feng and Winful (2001) inferred that for a Gaussian beam a phase shift would result as

$$\psi(x_3) = \int_0^{x_3} \left(\overline{k_1^2} + \overline{k_2^2}\right) dx_3 = \left(\frac{1}{2} + \frac{1}{2}\right) \tan^{-1}\left(\frac{x_3}{x_3^R}\right) \tag{2.2}$$

For $x_3 \to \infty$, then $\psi(x_3) = \pi/2$ ($\pi/4$ for each lateral dimension). For a Gaussian beam, this phase shift is progressive from 0 to $\pi/2$ from the beam waist for $0 < x_3 < \infty$. In 2-D ribbon beams, this phase shift goes from 0 to $\pi/4$. In Huygens-Fresnel integrals of a wavefront in terms of secondary wavelets, a $\pi/2$ phase shift is also required between the incident wavefront and the diverging secondary wavelets. For $-\infty < x_3 < +\infty$, the Gouy phase results in a phase shift of π for a wave going through a focus and for Gaussian beams this is progressive.

Paraxial Gaussian beams in inhomogeneous media can be described by dynamic ray tracing with complex initial conditions along a real ray, and this provides a major computational advantage for the calculation of high-frequency Gaussian beams in smoothly varying media. Overviews of paraxial Gaussian beams using dynamic ray tracing are given by Kravtsov and Berczynski (2007), Popov (2002), Cerveny (2001) and aren't discussed further here.

3. Bessel Beams in Homogeneous media. In the 1980's there was an interest in non-diffracting beam solutions including focus wave modes (Brittingham, 1983), exact wave solutions

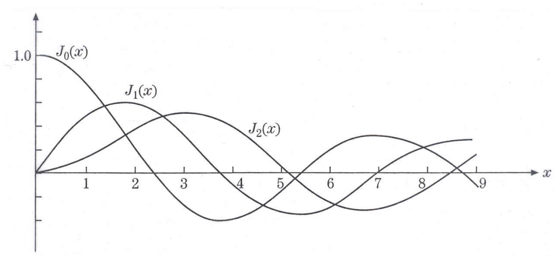

FIG. 3. *Cylindrical Bessel functions of different orders (from Weber and Arfken, 2004).*

with complex source locations (Ziolkowski, 1985), and even solutions called electromagnetic missiles (Wu, 1985). Durnin (1987) and Durnin et al. (1987) showed that Bessel beams can propagate without change of shape to a large range in free space. These types of beams were described earlier, for example by Stratton (1941, pp. 356), but because of their infinite energy (like plane waves), did not attract much interest at the time. To derive these solutions, consider the scalar wave equation

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{3.1}$$

A trial solution is used of the form

$$u(x,t) = f(\rho)\mathrm{e}^{\mathrm{i}(k_3 x_3 - \omega t)} \tag{3.2}$$

where $\rho = (x_1^2 + x_2^2)^{1/2}$ is the transverse distance to x_3, and the lateral shape $f(\rho)$ is preserved with distance x_3. Substituting this trial solution into the wave equation results in

$$\rho^2 \frac{d^2 f(\rho)}{d\rho^2} + \rho \frac{df(\rho)}{d\rho} + \rho^2\left(k^2 + k_3^2\right) f(\rho) = 0 \tag{3.3}$$

where $k^2 = \omega^2/v^2$. Recall Bessel's equation, (for example Weber and Arfken (2004), pp. 590)

$$x^2 \frac{dJ_\nu(x)}{dx^2} + x\frac{J_\nu(x)}{dx} + \left(x^2 - \nu^2\right) J_\nu(x) = 0 \tag{3.4}$$

where $J_\nu(x)$ is a cylindrical Bessel function of order ν. Figure 3 shows cylindrical Bessel functions of several orders. For equation 3.2, $f(\rho) = J_0(k_\rho \rho)$ where $k_\rho^2 = k^2 - k_3^2$. Therefore, a solution to the wave equation in free space that doesn't change lateral shape with distance is

$$u(x,t) = J_0(k_\rho \rho)\mathrm{e}^{\mathrm{i}(k_3 x_3 - \omega t)} \tag{3.5}$$

where $k^2 = \omega^2/v^2 = k_1^2 + k_2^2 + k_3^2 = k_\rho^2 + k_3^2$ with $k_\rho = k\,\sin\theta$ and $k_3 = k\,\cos\theta$. Then,

$$u(x,t) = J_0(k\sin\,\theta\rho)\mathrm{e}^{\mathrm{i}(k\,\cos\theta\, x_3 - \omega t)} \tag{3.6}$$

for some angle θ. In fact, Equations 3.5 and 3.6 represent "conical waves" which are analogous to plane wave solutions in cylindrical coordinates. For $\theta = 0$, the solution reduces to a plane wave traveling in the x_3 direction.

To show that this solution can be described in terms of conical waves, the Bessel function can be written as

$$J_0(\rho) = \frac{1}{2\pi}\int_0^{2\pi} d\alpha \mathrm{e}^{\mathrm{i}\rho\,\cos\alpha} \tag{3.7}$$

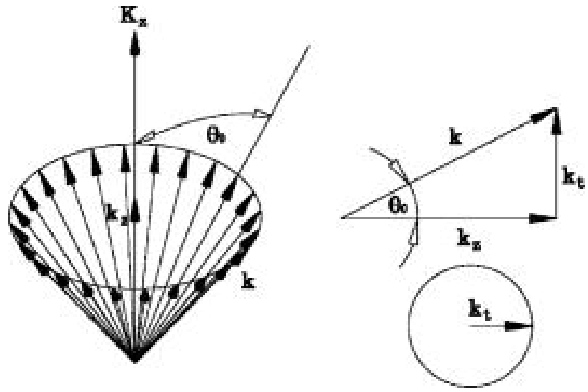

FIG. 4. *Wavenumber vectors making up a non-diffracting Bessel beam. All the plane waves in the Bessel beam have the same inclination angle θ_0 with respect to the propagation axis k_z (from Lopez-Mariscal et al., 2007).*

Let $\alpha = \phi - \phi'$, then $\cos\alpha = \cos(\phi - \phi') = \cos\phi\cos\phi' + \sin\phi\sin\phi'$. Also, let $x_1 = \rho\cos\phi'$ and $x_2 = \rho\sin\phi'$, then

$$J_0(\rho) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \mathrm{e}^{\mathrm{i}(\cos\phi\, x_1 + \sin\phi\, x_2)} \tag{3.8}$$

Now,

$$u(x,t) = J_0(k\sin\theta\rho)\mathrm{e}^{\mathrm{i}(k\cos\phi\, x_3 - \omega t)} \tag{3.9}$$

and,

$$u(x,t) = \int_0^{2\pi} d\phi \mathrm{e}^{\mathrm{i}(k\sin\theta\cos\phi\, x_1 + k\sin\theta\sin\phi\, x_2 + k\cos\theta\, x_3 - \omega t)} \tag{3.10}$$

This then equals

$$u(x,t) = \int_0^{2\pi} d\phi \mathrm{e}^{\mathrm{i}\overrightarrow{k}\cdot\overrightarrow{x} - \omega t} \tag{3.11}$$

where the integrand is now in the form of plane waves with $\overrightarrow{k} = \frac{\omega}{k}(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T$. Equation 3.11 defines a cone of plane waves normals with respect to the x_3 axis. In seismology, these are sometimes call conical waves, where the angle θ gives the opening angle of the cone. This cone of plane wave normals creates a Bessel beam which has a lateral cross-section which is invariant with distance, and thus is a diffraction free beam solution of the wave equation in free space (Figure 4).

The transverse cross-section is a Bessel function and and for intensity this is illustrated in Figure 5. Like a plane wave, Bessel beams have infinite energy and propagate in a diffraction free manner. In optics, a Bessel beam can be formed in several ways. Durnin et al. (1987) used an annular aperture followed by a lens to make plane waves. Bessel beams can also be formed by a so-called axicon lens (McGloin and Dholakia, 2005). In each of these cases, the diffraction free range x_3^{max} is limited practically by the size of the lens R, where $x_3^{max} = \frac{R}{\tan\theta}$ and θ is the open angle of the beam. Figure 6 shows a kaleidoscope pattern of a non-diffracting beam formed from a discrete set of a finite number of plane waves (from Bouchal, 2003).

Bessel beams are free-wave mode solutions in a cylindrical coordinate system, and therefore can be used to decompose other cylindrically symmetric wavefields. For example, a spherical wave can be either decomposed into plane waves as

$$\frac{\mathrm{e}^{\mathrm{i}\omega R/v}}{R} = \frac{\mathrm{i}}{2\pi}\int_{\infty}^{\infty} dk_1 dk_2 \frac{1}{k_3}\mathrm{e}^{\mathrm{i}(k_1x_1 + k_2x_2 + k_3|x_3|)} \tag{3.12}$$

FIG. 5. *The transverse intensity pattern of a zero-order Bessel beam (from Bouchal, 2003).*

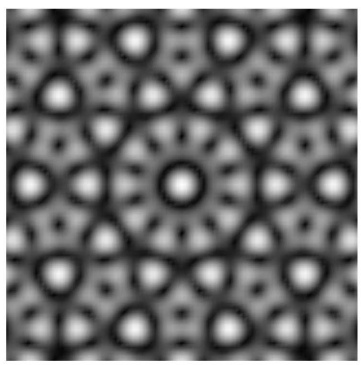

FIG. 6. *A kaleidoscope of nondiffracting beam patterns can be obtained as a discrete set of a finite number of plane waves. This shows the case for $N = 5$ plane waves (from Bouchal, 2003).*

where $k_3 = (\omega^2/v^2 - k_1^2 - k_2^2)^{1/2}$ with $\mathrm{Im}(k_3) > 0$ and $\mathrm{Re}(k_3) > 0$, and this is called the Weyl integral (Aki and Richards, 1980; Chew, 1990). Also,

$$\frac{\mathrm{e}^{\mathrm{i}\omega R/v}}{R} = \mathrm{i}\int_0^\infty dk_\rho \frac{k_\rho}{k_3} J_0(k_\rho \rho)\mathrm{e}^{\mathrm{i}k_3|x_3|} \tag{3.13}$$

and this is called the Sommerfeld integral (Aki and Richards, 1980; Chew, 1990), which is a decomposition of a spherical wave into conical waves or Bessel beams.

As a comparison of Gaussian beams and Bessel beams, Gaussian beams concentrates the energy, but are diffracting. In contrast, Bessel beams have a transverse pattern which is stationary in x_3 and therefore are non-diffracting, but the energy is not all concentrated along the central axis. Figure 7 compares a Gaussian beam with a finite-aperture limited-diffraction Bessel beam (from Salo and Friberg, 2008). The central maximum of the diffraction free beam has the same width as the beam-waist of the Gaussian beam. The diffraction free range of the Bessel beam is ultimately limited by the size of the initial lens or other aperture as described earlier.

A question of "superluminal" behavior of beam solutions in free space (or faster than light speed or medium speed in acoustics) was asked by Mugnai et al. (2000). Recall that a Bessel beam can

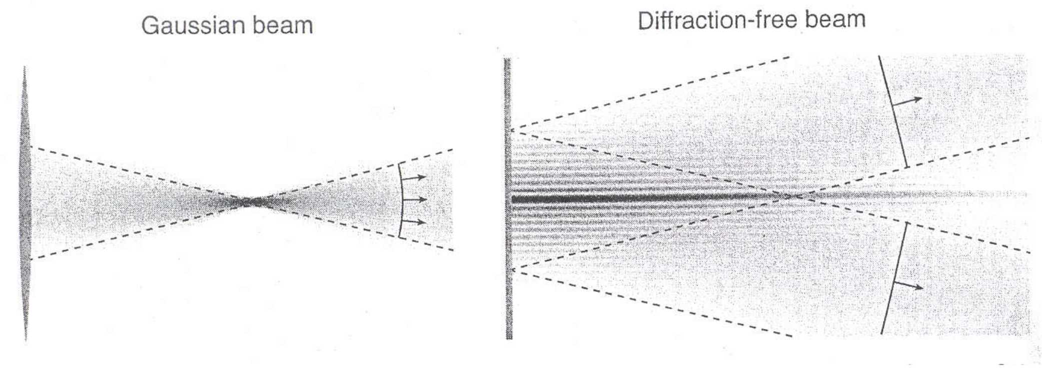

FIG. 7. *A Gaussian beam and a finite-aperture limited-diffraction Bessel beam are compared. The maximum of the central maximum of the Bessel beam has the same width as the beam waist of the Gaussian beam (from Salo and Friberg, 2008).*

be written as

$$u(x,t) = J_0(k \sin\theta\, \rho)\mathrm{e}^{\mathrm{i}(k\, \cos\theta\, x_3 - \omega t)} \tag{3.14}$$

so the "velocity" of the Bessel beam v^{bb} in the x_3 direction is

$$k_3\, x_3 = k\, \cos\theta\, x_3 = \frac{\omega}{v} \cos\theta\, x_3 = \frac{\omega x_3}{v^{bb}} \tag{3.15}$$

where $v^{bb} = \frac{v}{\cos\theta}$. When $\theta > 0$, then $v^{bb} > v$ where v is the medium speed. So one can ask, can a Bessel beam really go faster than the medium velocity, and the answer is yes. But, then one can ask, can a EM wave go faster than light speed, and again the answer is yes. The question then is, can energy for a Bessel beam go faster than the medium speed (or the speed of light), and the answer is no. It turns out that v^{bb} is an apparent phase speed in the x_x direction.

Since a Bessel beam is made up of plane waves, each traveling at an angle θ to the x_3 axis, it is most straightforward to look at the individual plane waves making up the Bessel beam. The $\overrightarrow{k}$ vector for a plane wave is in the direction of the wavefront normal and $k_i = \frac{\omega}{v} s_i$ where $\overrightarrow{s}$ is the unit normal to the plane wave. The phase velocity is then $v_i^p = \frac{\omega}{k_i} = \frac{v}{s_i}$. For s_i in the $x_1 - x_3$ plane, then $\overrightarrow{s} = (\sin\theta, 0, \cos\theta)$ for some angle θ from x_3 direction. Then, $v_1^p = \frac{v}{\sin\theta}$ and $v_3^p = \frac{v}{\cos\theta}$. For $\theta = 0$, then $v_3^p = v$ the phase speed of the medium. For $\theta > 0$, then $v_3^p > v$. As an example, for a wave hitting a beach, the apparent speed of the wave along the direction of the beach can be large, and even infinite if $\theta = 90$ degrees when all points on the beach are hit at the same time.

However, energy does not travel with the phase velocity, but rather at the group velocity for a non-attenuating medium (Cerveny, 2001) where

$$v_i^g = \frac{d\omega}{dk_i} \tag{3.16}$$

Now $s_i k_i = s_i \frac{\omega}{v} s_i = \frac{\omega}{v}$ since $s_i s_i = 1$. Then $\omega = v s_i k_i$. So, $v_i^g = \frac{d}{dk_i}(v s_j k_j) = v s_i$. Thus, for an individual plane wave traveling at an angle θ to the x_3 axis, then $v_1^g = v \sin\theta$ and $v_3^g = v \cos\theta$. For $\theta = 0$, $v_3^g = v = v_3^p$ in the direction of wavefront travel. For $\theta > 0$, then $v_3^g < v$ and $v^p > v$. Therefore, the apparent v^g in the x_3 direction is less than or equal to the medium speed. Thus, for an oblique direction to the plane wave direction, group velocities cannot go faster than v. Nonetheless, apparent phase velocities can go faster than v, and this is regularly used and measured in seismological applications using arrays. As a check on this, a relation for plane waves in both isotropic and anisotropic, non-attenuating media is

$$\overrightarrow{v^g} \cdot \overrightarrow{p} = 1 \tag{3.17}$$

where $\overrightarrow{p} = \overrightarrow{s}/v$ is the slowness vector (Cerveny, 2001, Eqn. 2.2.67). Using the relations above for the group velocity then $(vs_i)(s_i/v) = s_i s_i = 1$.

Since Bessel beams are made up of a cone of plane waves all traveling obliquely with the same angle θ with respect to the x_3 axis, then $v^{bb} = v/\cos\theta$ is the phase velocity of the Bessel beam package traveling in the x_3 direction. Although this velocity is greater than the medium velocity v, there are no "faster than light" difficulties, but rather only confusion between phase, group and energy velocities.

In the physics literature there was some early confusion on different definitions of beam velocity since $k_3 = k\cos\theta = \frac{\omega}{v}\cos\theta$ and some researchers inferred $v_3^g = \frac{d\omega}{dk_3} = \frac{k_3 v}{\cos\theta} = v/\cos\theta$. So it was incorrectly concluded that v_3^g was greater than v, the same as v^p for a Bessel beam. (see for example, McDonald, 2000). Nonetheless, an energy velocity for a Bessel beam was derived, and this was thought to be different from both the phase and group velocity. For example, Sauter and Paschke (2001) obtained an energy velocity of $v^e = v\cos\theta$ and concluded that energy for a Bessel beam did not go faster than light. However, as shown above this is also the group velocity in a non-attenuating medium for the individual plane waves making up the Bessel beam. As a result of these controversies, different definitions of signal velocities were re-assessed (Milonni, 2005), and there was also a renewed interest in classic work on wave speeds, such as that of Brillouin (1960).

In addition to Bessel beams in free space, a number of "paradoxes" were found in "exotic media" where faster than light propagation have been inferred in recent years. For example, Wang et al. (2000) found superluminal light propagation in gain-assisted media. In examples like these, further explanations are required for large apparent velocities along the beam axis. Also, there have been inferences of ultra-slow light propagation in special media, for example by Vertergaard et al. (1999). A number of examples of fast and slow light have been summarized by Milonni (2005). How these results will ultimately be interpreted and if they have potential applications in seismology is yet to be determined.

As an example of multi-frequency pulsed-signals, Heyman and Felsen (2001) investigated Gaussian beams and pulsed-beam dynamics using complex-source and complex-spectrum formulations. For pulsed Bessel beams, researchers have found so-called "X-waves" which travel in the shape of an "X" in the x_3 direction. In acoustics, these were described and observed by Lu and Greenleaf (1992a,b) from work performed at the Mayo Clinic (see also, Lu and Greenleaf, 1994; Lu, 2008). For light propagation, Figure 8 shows an example of Bessel X-wave propagation from Saari and Reivelt (1997) where the simulated result is shown on the left panel and experimental results are shown on the right panel.

One way to think about pulsed Bessel beam signals is to just consider two crossing plane waves in 2D. Figure 9 shows an X-pulse formed from two crossing plane waves at two times moving in the vertical direction (from Sauter and Paschke, 2001). The overall shape of the pulse moves in the vertical x_3 direction without change of shape. In an actual X-wave, this would result in pulsed conical waves with plane waves over the entire cone of wavenumbers. The center "X" moves in the vertical x_3 direction at a speed of $\frac{v}{\cos\theta} > v$ (the center of the "X" moves at the phase velocity). An analogy used for this is of a closing scissors where each metal piece of the scissors is moving slowly, but the crossing point can move very fast. Also, since different parts of the wavefronts make up the center of the "X" at different times and distances, the waveform has a "self-healing" property to local perturbations along the center of the "X" of the wave. Practically, these pulsed X-waves must be passed through an aperture and this will ultimately limit the propagation distance of the wave before it diffuses.

Finally, a different phenomena, but a relative to a Bessel beam, is the "spot of Arago" or sometimes call "Poisson's spot". In 1818, Fresnel presented his paper on the diffraction theory of light. This was at a time when Newton's corpuscular theory of light was the preferred model. Attempting to invalidate Fresnel's conclusion, Poisson, a member of the examining committee, predicted that if you put a small obstacle in front of a source of light, a spot of light would appear at the center of the shadow. Poisson claimed that this would violate common sense and would then disprove Fresnel's theory. However, the prediction was experimentally tested and verified by Arago,

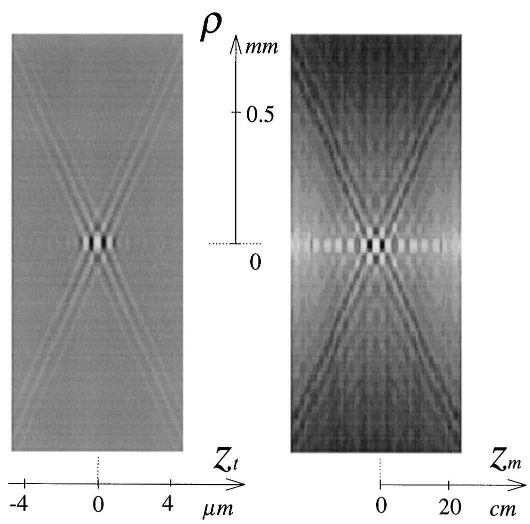

FIG. 8. *Bessel X-wave for a light wave where the left panel shows the simulated results and the right panel shows the experimental results. Here the direction of propagation is horizontal (from Saari and Reivelt, 1997)*

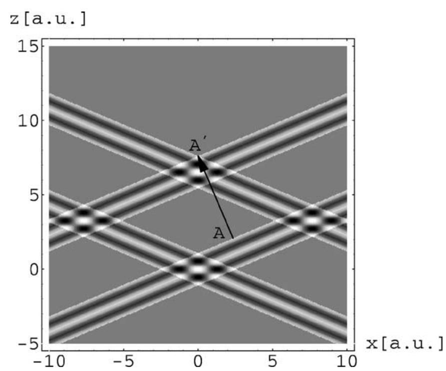

FIG. 9. *An X-pulse formed from two crossing plane waves at two times moving in the vertical direction. The center of the "X" at different times comes from different parts of the two plane waves (from Sauter and Paschke, 2001).*

another member of the examining committee, thus validating the wave theory of light (see, Harvey and Forgham, 1984).

4. Conclusions. In this overview, Gaussian beams and Bessel beams have been described. Both can be decomposed into plane waves. For a Gaussian beam which is localized but undergoes diffraction, the spatial transform of Equation 2.1 for $x_3 = 0$ is

$$S(k_\rho, \omega) = 2^{1/2} \mathrm{e}^{-W_0^2 k_\rho^2/4}\, \delta(\omega - \omega_0) \tag{4.1}$$

where W_0 is the initial beam width and $k_\rho^2 = k_1^2 + k_2^2$ at a single frequency ω_0. For a Bessel beam which is ideally non-diffracting, then

$$S(k_\rho, \omega) = \frac{\delta(k_\rho - \frac{\omega}{v}\sin\theta)}{k_\rho} \delta(\omega - \omega_0) \tag{4.2}$$

where this describes a cone in wavenumber space at a single frequency ω_0. Although for a given opening angle, a Bessel beam excludes the central angles required to form a Gaussian beam, a cylindrically symmetric Gaussian beam can be decomposed into Bessel beams of different angles θ in a similar fashion as spherical waves decomposed into conical waves.

Acknowledgements. This work was supported in part by the National Science Foundation and the Air Force Geophysics Laboratory and partly by the members of the Geo-Mathematical Imaging Group (GMIG) at Purdue University.

REFERENCES

[1] Aki, K. and P. Richards, (1980) *Quantitative Seismology*, Freeman, San Francisco.
[2] Bateman, H., (1915) *Electrical and Optical Wave Motion*, Cambridge Univ. Press.
[3] Bouchal, Z., (2003) *Nondiffracting optical beams: physical properties, experiments, and applications*, Czech J. Phys., 53, 537-578.
[4] Brillouin, L.,(1960) *Wave Propagation and Group Velocity*, Academic Press, New York.
[5] Brittingham, J.N., (1983) *Focus wave modes in homogeneous Maxwell's equations: transverse electric mode*, J. Appl. Phys., 54, 1179-1189.
[6] Cerveny, V., (2001) *Seismic Ray Theory*, Cambridge University Press.
[7] Chew, W.C., (1990) *Waves and Fields in Inhomogeneous Media*, Van Nostrand Reinhold.
[8] Courant, R. and D. Hilbert, (1966) *Methods of Mathematical Physics*,Vol. 2, Wiley, New York, pp. 760.
[9] Dechamps, G. A., (1987) *Gaussian beams as a bundle of complex rays*, Electron. Lett., 7, 684-685.
[10] Durnin, J., (1987) *Exact solutions for nondiffracting beams: I. the scalar theory*, J. Opt. Soc. Am. A, 4, 651-654.
[11] Durnin, J., J. J. Miceli adn J.H. Eberly, (1987) *Diffraction-free beams*, Phy. Rev. Lett., 58, 1499-1501.
[12] Felsen, L.B., (1976), *Complex-source-point solutions of the field equations and their relation to the propagation and scattering of Gaussian beams*, Symp. Math., 18, 39.
[13] Feng, S. and H.G. Winful, (2001) *Physical origin of the Gouy shift*, Opt. Lett., 26, 485-487.
[14] Flammer, C., (1957) *Speroidal Wave Functions*, Stanford Press.
[15] Harvey, J.E. and J.L. Forgham, (1984) *spot of Arago: new relevance for an old phenomenon*, Am. J. Phys., 52, 243-247.
[16] Heyman, E. and L.B. Felsen,(2001) *Gaussian beam and pulsed-beam dynamics complex-source and complex-spectrum formulations within and beyond paraxial asymptotics*, J. Opt. Soc. Am. A, 18, 1588-1611.
[17] Kravtsov, Yu A. and P. Berczynski, (2007) *Gaussian beams in inhomogeneous media: a review*, Studia Geophys. et Geod., 51, 1-36.
[18] Lopez-Mariscal, C. and J.C. Gutierrez-Vega, (2007) *The generation of nondiffracting beams using inexpensive computer generated holograms*, Am. J. Phys., 75, 36-42.
[19] Lu, J.Y., (2008) *Ultrasonic imaging with limited-diffraction beams*, in Localized Waves (eds. H.E. Hernandez-Figueroa, M. Zamboni-Rached and E. Recami) Wiley Interscience, pp. 97-128.
[20] Lu, J.Y. and J.F. Greenleaf, (1992a) *Nondiffracting X-waves: exact solution to free-space scalar wave equation and their finite aperture realizations*, IEEE Trans. Ultrason. Ferroelectr. Freq. Control, 39(1), 19-31.
[21] Lu, J.Y. and J.F. Greenleaf, (1992b) *Experimental verification of nondiffracting X-waves*, IEEE Trans. Ultrason. Ferroelectr. Freq. Control, 39(3), 441-446.
[22] Lu, J.Y. and J.F. Greenleaf, (1994) *Biomedical ultrasound beam forming*, Ultrasound Med. Bio., 20(5), 403-428.
[23] Lunardi, J.T., (2001) *Remarks on Bessel beams, signals and superluminality*, Phys. Lett. A, 291, 66-72.
[24] McDonald, K.T., (2002) *Gaussian laser beams via oblate spheroidal waves*, *www.hep.princeton.edu/~mcdonald/examples/* .
[25] McDonald, K.T., (2000) *Bessel beams*, *www.hep.princeton.edu/~mcdonald/examples/* .
[26] McGloin, D. and K. Dholakia, (2005) *Bessel beams: diffraction in a new light*, Contemporary Physics, 46, 15-28.
[27] Milonni, P.W., (2005) *Fast light, slow light and left-handed light*, Institute of Physics Publ. Bristol.
[28] Mugnai, D., A. Ranfagni and R. Ruggeri, (2000) *Observation of superluminal behaviors in wave propagation*, Phys. Rev. Lett., 84, 4830-4833.
[29] Popov, M.M., (2002) *Ray Theory and Gaussian Beam Method for Geophysicists*, Lecture Notes, Univ. of Bahia, Salvador, Brazil.
[30] Recami, E., M. Zamboni-Rached adn H.E. Hernandez-Figuerao, (2008) *Localized waves: a historical and scientific introduction*, in Localized Waves (eds. H.E. Hernandez-Figueroa, M. Zamboni-Rached and E. Recami) Wiley Interscience, pp. 1-41.
[31] Saari, P. and K. Reivelt, (1997) *Evidence of X-shaped propagation-invariant localized light waves* Phys. Rev. Lett., 79, 4135-4138.
[32] Salo, J. and A.T. Friberg, (2008) *Propagation-invariant fields: rotationally periodic and anisotropic non-diffracting waves*, in Localized Waves (eds. H.E. Hernandez-Figueroa, M. Zamboni-Rached and E. Recami) Wiley Interscience, pp. 129-157.
[33] Sauter, T. and F. Paschke, (2001) *Can Bessel beams carry superluminal signals?*, Phys. Lett. A, 285, 1-6.
[34] Stratton, J.A., (1941) *Electromagnetic Theory*, McGraw-Hill, New York, pp. 356.
[35] Stratton, J.A., (1956) *Spheroidal Wave Functions*, Wiley, New York.
[36] Siegman, A. E. , (1986) *Lasers*, University Science Books.
[37] Vertergaard Hau, L., S.E. Harris, Z. Dutton, and C.H. Behroozi,(1999) *Light speed reduction to 17*

metres per second in an ultracold atomic gas, Nature, 397, 594-598.
[38] Wang, L.J., A. Kuzmich adn A. Dogariu, (2000) *Gain-assisted superluminal light propagation*, Nature, 406, 277-279.
[39] Weber, H.J. and G.B. Arfken, (2004) *Essential Mathematical Methods for Physicists*, Elsevier.
[40] Wu, T.T., (1985) *Electromagnetic missiles*, J. Appl. Phys., 57, 2370-2373.
[41] Ziolkowski, R.W., (1985) *Exact solutions of the wave equation with complex source locations*, J. Math. Phys., 26, 861-863.