12/08

1. ETEMADI'S INEQUALITY

2. WEAK CONVERGENCE OF EMPIRICAL DIST.

1.

THEOREM $\{X_1, \dots, X_n\}$ ARE INDEPENDENT R.V.'S. & $S_k := \sum_{i=1}^{k} X_i$. THEN $\forall\ x > 0$,

$$\mathbb{P}\left(\max_{1 \le k \le n} |S_k| \geq 3x \right)$$

$$\le 2\,\mathbb{P}\left(|S_n| \geq x \right) + \max_{1 \le k \le n} \mathbb{P}\left(|S_k| \geq x \right)$$

$$\le 3 \cdot \max_{1 \le k \le n} \mathbb{P}\left(|S_k| \geq x \right).$$

PROOF:

DEFINE:

$$A := \left\{ \max_{1 \le k \le n} |S_k| \geq 3x \right\}$$

FOR $1 < k \le n$

$$A_k := \left\{ \max_{1 \le j \le k-1} |S_j| < 3x \right\}$$

$$\cap \left\{ |S_k| \geq 3x \right\}$$

& $$A_1 := \left\{ |S_1| \geq 3x \right\}$$

OBSERVE:

- $$A = \bigcup_{k=1}^{n} A_k .$$

- A_k ARE DISJOINT

CONSIDER:

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}\left(A \cap \{ |S_n| \geq x \} \right) \\ &\quad + \mathbb{P}\left(A \cap \{ |S_n| < x \} \right) \\ &\leq \mathbb{P}\left(|S_n| \geq x \right) \\ &\quad + \mathbb{P}\left(\bigcup_{k=1}^{n} \left\{ A_k \cap \{ |S_n| < x \} \right\} \right) \end{aligned}$$

$$\Rightarrow \quad \mathbb{P}(A) \leq \mathbb{P}(|S_n| \geq x)$$

$$+ \sum_{k=1}^{n} \mathbb{P}\left(A_k \cap \{|S_n| < x\}\right)$$

OBSERVE:

BY Δ^{le} INEQUALITY:

$$|S_k| \leq |S_n - S_k| + |S_n|$$

SUPPOSE : $\{|S_n| < x\}$ & A_k HOLD.

$$\Rightarrow \quad |S_n - S_k| \geq |S_k| - |S_n|$$

$$\geq 2x .$$

THAT IS:

$$A_k \cap \{ |S_k| < x \}$$

$$\subset \quad A_k \cap \{ |S_n - S_k| \geqslant 2x \}$$

$$\therefore \quad \mathbb{P}(A) \leq \mathbb{P}(|S_n| \geqslant x)$$

$$+ \sum_{k=1}^{n} \mathbb{P}\left(A_k \cap \{ |S_n - S_k| \geqslant 2x \} \right).$$

BY DEFINITION OF S_k,

$$A_k \perp \{ |S_n - S_k| \geq 2x \}.$$

$$\Rightarrow \mathbb{P}(A) \leq \mathbb{P}(|S_n| \geq x)$$

$$+ \sum_{k=1}^{n} \mathbb{P}(A_k)\mathbb{P}(\{ |S_n - S_k| \geq 2x \})$$

$$\leq \mathbb{P}(|S_n| \geq x)$$

$$+ \mathbb{P}(A) \cdot \max_{1 \leq k \leq n} \mathbb{P}(|S_n - S_k| \geq 2x)$$

FINALLY:

$$|a-b| \geq 2x$$

$$\Rightarrow \quad |a| > x \quad \text{OR} \quad |b| > x.$$

$$\therefore \ \mathbb{P}(A) \leq \mathbb{P}(|S_n| \geq x) + \max_{1 \leq k \leq n} \mathbb{P}\left(\{|S_n| > x\} \cup \{|S_k| > x\}\right)$$

$$\leq 2\,\mathbb{P}(|S_n| \geq x) + \max_{1 \leq k \leq n} \mathbb{P}(|S_k| > x)$$

$$\leq 3 \cdot \max_{1 \leq k \leq n} \mathbb{P}(|S_k| > x).$$

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2. WEAK CONVERGENCE OF EMPIRICAL DISTRIBUTION:

THEOREM A: IF $\{\xi_n\}$ ARE IID UNIFORM WITH SUPPORT $[0,1]$ AND

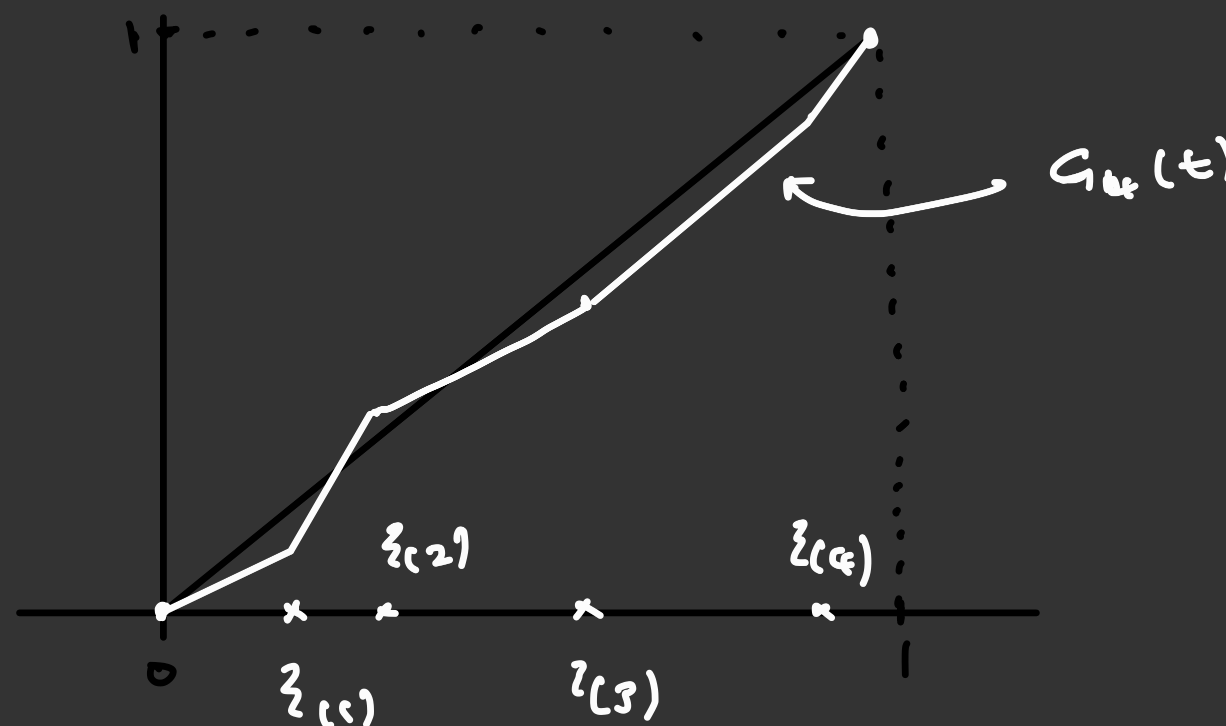
$$Z_n(t) := \sqrt{n}\left(G_n(t) - t \right)$$

WHERE: $G_n(t)$ IS THE DISTRIBUTION FUNCTION CORRESPONDING TO A UNIFORM DIST. OF $\frac{1}{n+1}$ MASS ON INTERVAL $[\xi_{(i-1)}, \xi_{(i)}]$, $\xi_{(0)} = 0$, $\xi_{(n+1)} = 1$, $(\xi_{(1)}, \dots, \xi_{(n)})$ ARE THE ORDERED STATISTICS OF $(\xi_1, \dots, \xi_n)$.

THEN:

$$Z_n \Rightarrow_n W^{\circ},$$

WHERE W° IS THE SO-CALLED BROWNIAN BRIDGE PROCESS.



BROWNIAN BRIDGE :

GAUSSIAN PROCESS ON $[0,1]$ WITH THE FOLLOWING PROPERTIES:

1. GAUSSIAN MARGINALS $\forall\ t \in (0,1)$,

$$\mathbb{E}\, W^{\circ}_t = 0$$

$$\mathbb{E}\, W^{\circ 2}_t = t(1-t) \quad \forall\, t$$

$$W^{\circ}_1 = 0 = W^{\circ}_0. \quad (\text{a.s.})$$

2. COVARIANCE:

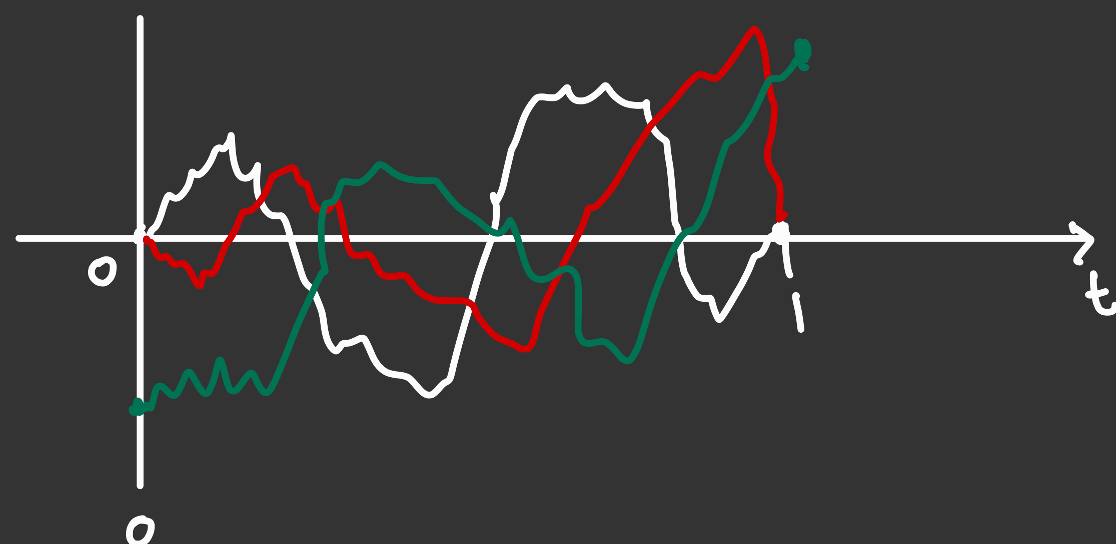
$$\mathbb{E}\, W^{\circ}_t W^{\circ}_s = s(1-t) \quad \forall\ s \leq t.$$

ALSO KNOWN:

$$W_t^0 = W_t - t W_1 \qquad 0 \le t \le 1.$$

$$\mathbb{P}(W^0 \in A) = \mathbb{P}(W \in A \mid W_1 = 0).$$

"TIED DOWN" B.M.



GLIVENKO - CANTELLI THEOREM

$$F_n(t) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{\xi_i \le t\}} \quad \forall\, t \in [0,1]$$

THEN:

$$F_n \rightrightarrows_n F \qquad \text{as} \quad n \to \infty$$

$F(t) = t \quad \forall t \in [0,1]$

RECALL: $F_n(1) = F(1) = 1$

$$\sqrt{n}\left(F_n(1) - F(1)\right) = 0$$

A USEFUL TIGHTNESS CRITERION:

(TH. 12.3 IN BILL 1968)

THEOREM B: $\{X_n\} \subset C$ IS TIGHT IF:

(i) $\{X_n(0)\}$ IS TIGHT

(ii) $\exists$ CONSTANTS $\gamma \geq 0$ & $\alpha < 1$ & A NON-DECREASING FUNCTION F ON $[0,1]$ S.T.

$$P\Big(|X_n(t_2) - X_n(t_1)| \geq \lambda \Big) \leq \frac{1}{\lambda^\gamma} |F(t_2) - F(t_1)|^\alpha$$

HOLDS $\forall\ t_1, t_2$ & n $\forall\ \lambda > 0$.

HAHN (1975)

A STRONGER CONDITION:

$$E\,|X_n(t_2) - X_n(t_1)|^\gamma \leq |F(t_2) - F(t_1)|^\alpha.$$

PROOF OF THEOREM A:

(i) CONVERGENCE OF THE FDD'S:

FIRST CONSIDER A SINGLE $t \in [0,1]$.

$$Z_n(t) = \sqrt{n}\left(G_n(t) - t\right)$$

BY DEMOIVRE-LAPLACE:

$$Z_n(t) \Rightarrow \mathcal{N}(0, t(1-t)).$$

FOR THE GENERAL CASE: $t_0 = 0 < t_1 < t_2 < \dots < t_d \leq 1$

$$\left(Z_n(t_1), Z_n(t_2) - Z_n(t_1), \dots, Z_n(t_d) - Z_n(t_{d-1})\right)$$

$$\Rightarrow \mathcal{N}(0, \Sigma_d) \qquad \sigma_{ik} = (t_i - t_{i-1})(t_k - t_{k-1})$$

THIS FOLLOWS FROM THE MULTINOMIAL CLT.

$\left(Z_n(t_1), Z_n(t_2), \dots, Z_n(t_d) \right)$ CONVERGES TO THE

"RIGHT" GAUSSIAN R.V. WITH COVARIANCE

MATRIX $\tilde{\Sigma}_d$ WITH ENTRIES $\tilde{\sigma}_{ik} = t_i(1-t_k)$.

(ii) TIGHTNESS: VERIFY THEOREM B.

$$\mathbb{E} \left| Z_n(t_2) - Z_n(t_1) \right|^4$$

$$= n^2 \mathbb{E} \left| \left(G_n(t_2) - G_n(t_1) \right) - (t_2 - t_1) \right|^4$$

$$\leq 6 \, |t_2 - t_1|^2 .$$

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FINALLY: CONSIDER $F_n(t) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{\xi_i \le t\}}$.

IT CAN BE SEEN:

$$\left| F_n(t) - G_n(t) \right| \le \frac{1}{n} \qquad 0 \le t \le 1.$$

LET, $Y_n(t) := \sqrt{n}\left(F_n(t) - t\right)$. THEN

$$\sup_t \left| Y_n(t) - Z_n(t) \right| \le \frac{1}{\sqrt{n}}.$$

IF F IS NOT UNIFORM, THEN,

$$Y_n \Rightarrow_n W^\circ \circ F. \qquad \#$$

DONALD FEIGIN

$$Z(t) = F(t) + \frac{1}{\sqrt{n}} W(t)$$