A metric space (S, ρ) is defined by a space S and a metric ρ.

The metric ρ is a function $\rho : S \times S \longrightarrow \mathbb{R}^+$ such that $\forall\ x, y, z \in S$

(i) $\rho(x, y) = 0$ iff $x = y$

(ii) $\rho(x, y) = \rho(y, x)$

(iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

Examples

(i) Euclidean plane $\mathbb{R}^2$

For each $x, y \in \mathbb{R}^2$

$$\rho(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

(ii) Function Space $C[a, b]$

(each "point" is a real-valued function on $[a, b]$)

$$\rho(x, y) = \sup_{t \in [a,b]} |x(t) - y(t)|$$

For a subset $A \subseteq S$, we denote

(i) A^{-} for closure;

(ii) A° for interior;

(iii) $\partial A = A^{-} - A^{\circ}$ for boundary.

Also,

$$\rho(x, A) = \inf_{y \in A} \{ \rho(x, y) \}$$

$\rho(\cdot, A)$ is uniformly continuous.

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The metric ρ is said to be finer than ρ' if the open sets generated by ρ subsume those of ρ', that is,

$$\mathcal{O} \supseteq \mathcal{O}'$$

where $\mathcal{O}, \mathcal{O}'$ are the classes of open sets generated by ρ and ρ'.

Two metrics ρ_1 and ρ_2 are said to be equivalent if each is finer than the other.

(S, ρ_1) and (S, ρ_2) are then homeomorphic.

The space S is separable if it has a countable dense subset.

The following are equivalent.

(i) S is separable

(ii) S has a countable base.

(iii) Each open cover of each subset of S has a countable sub cover.

The space S is complete if Cauchy sequences are convergent with limit in S, that is,

$$\sup_{i,j \geq n} \rho(x_i, x_j) \to 0$$

implies

$$\{x_j\} \to x \in S.$$

Separability is a topological property; completeness is not a topological property.

(Loosely, separability does not depend on the metric but completeness does.)

A set A is compact if any open cover of A can be replaced by a finite sub-cover.

Compactness is a topological property.

An ε-net for A is a set $\{x_k\}$ such that for each $x \in A$, $\exists$ x_k such that

$$\rho(x, x_k) < \varepsilon.$$

The set A is called totally bounded if for each $\varepsilon > 0$, it has a finite ε-net.

The following are equivalent.

(i) $\bar{A}$ is compact

(ii) Each sequence in A has a convergent subsequence.

(iii) A is totally bounded and $\bar{A}$ is complete.

Examples for Intuition

First, recall the concepts.

(i) Separability : the space S is separable if it has a countable dense subset.

(ii) Completeness : the space (S, ρ) is said to be complete if Cauchy sequences are convergent.

Examples for Intuition

First, recall the concepts.

(iii) Compactness: S is said to be compact if for each $\{x_k\} \subseteq S$ there exists $\{x_{k_j}\} \to x \in S$.

(iv) Separating Class: $\mathcal{S}' \subseteq \mathcal{S}$ is a separating class of sets in S if

$$P(A) = Q(A) \quad \forall A \in \mathcal{S}'$$
$$\Rightarrow P(A) = Q(A) \quad \forall A \in \mathcal{S}.$$

Examples for Intuition

① **Separable but not complete:**

the space $S = [1, \infty)$ is clearly separable, but not complete under

$$\rho(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right| .$$

$S = [1, \infty)$ is **not compact.**

(2) The space

$$\ell^\infty := \left\{ x : x = (\xi_1^x, \xi_2^x, \ldots) \right\};$$

$$c_x = \sup_n \{ |\xi_n^x| \} < \infty$$

is not separable;

however, the space

$$\ell^p := \left\{ x : x = (\xi_1^x, \xi_2^x, \ldots) \right\}$$

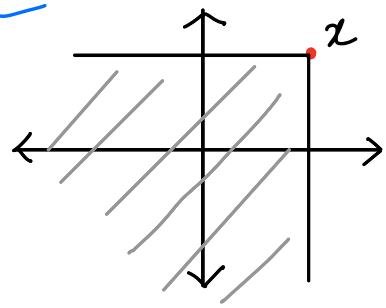
$$c_{p,x} = \sum_n |\xi_n^x|^p < \infty$$

is separable.

ℓ^∞ and ℓ^p are complete

(3) Define

$$\mathcal{G}(x) := \left\{ y \in \mathbb{R}^k : y_i \leq x_i, \ i \leq k \right\}$$



Consider

$$\mathcal{G} := \left\{ \mathcal{G}(x) : x \in \mathbb{R}^k \right\}.$$

We can show that $\mathcal{G}$ is a π-system that generates $\mathcal{B}(\mathbb{R}^k)$. Hence $\mathcal{G}$ is a separating class.

④ $\mathbb{R}^\infty$ is the space of sequences $x = (x_1, x_2, \dots)$ equipped with

$$\rho(x, y) = \sum_i \frac{b(x_i, y_i)}{2^i};$$

$$b(\alpha, \beta) = 1 \wedge |\alpha - \beta|.$$

$\mathbb{R}^\infty$ is separable and complete.

④ contd...

Define $\pi : \mathbb{R}^\infty \longrightarrow \mathbb{R}^k$

$$\pi_k(x) = (x_1, x_2, \dots x_k)$$

and the finite-dimensional class

$$\mathcal{R}_f^\infty := \left\{ \pi_k^{-1}(H),\ H \in \mathcal{B}(\mathbb{R}^k) \right\}$$

$\mathcal{R}_f^\infty$ is a π-system generating $\mathcal{B}(\mathbb{R}^\infty)$; hence a separating class.

(5) The space $C[0,1]$ of equipped with the metric

$$\rho(x,y) := \sup\left\{|x(t)-y(t)|,\ t\in[0,1]\right\}.$$

$\mathcal{C} := \sigma(C[0,1])$ is the Borel σ-field.

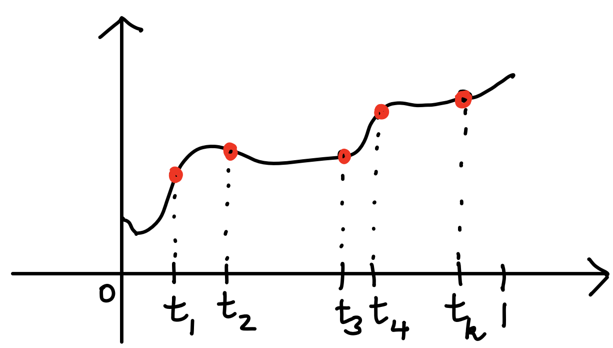
$C[0,1]$ is separable and complete but not compact.

⑤ contd...

Define

$$C_f := \left\{ \pi^{-1}_{t_1 \cdots t_k} H, \; H \in \mathcal{B}(\mathbb{R}^k) \right\}$$

where $\pi_{t_1 \cdots t_k}(x) = \big(x(t_1), \ldots, x(t_k)\big)$.



C_f is a π-system that generates $\mathcal{C}$ and is hence a separating class