Metric Space (S, ρ)

"universe" "metric"

Four Stipulations:

(1) $\rho : S \longrightarrow \mathbb{R}^+$

(2) $\rho(x, y) = 0$ iff $x = y$.

(3) $\rho(x, y) = \rho(y, x)$

(4) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

Examples of Metric Spaces

(i) Euclidean plane $\mathbb{R}^2$ with

$$\rho(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

(ii) Function space $C[a,b]$

$$\rho(x,y) = \max_{t\in[a,b]} |x(t) - y(t)|$$

(iii) Sequence space ℓ^∞

$$x = \{\xi_n^x, n \geq 1\}, \quad |\xi_n^x| \leq c_x < \infty$$

$$\rho(x,y) = \sup_n \{|\xi_n^x - \xi_n^y|\}$$

For the metric space (S, ρ) $\mathcal{S}$ denote the Borel σ-field.

For a probability measure P on $(S, \mathcal{S})$, we write Pf to mean $\int_S f \, dP$ for bounded, continuous real-valued f.

For probability measures P and P_n, we say "P_n converges weakly" to P and write

$$P_n \Longrightarrow P$$

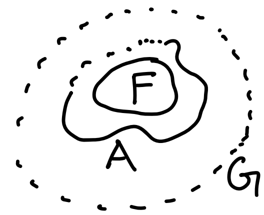
if

$$P_n f \longrightarrow P f \quad *$$

for $f \in \mathcal{BC}(S)$ where,

$$\mathcal{BC}(S) := \left\{ \begin{array}{l} \text{bounded continuous} \\ \text{real } f \text{ on } S \end{array} \right\}$$

Theorem 1.1



Every probability measure P on $(S, \mathcal{S})$ is regular, that is, for each $A \in \mathcal{S}$, $\exists$ closed $F \subseteq A$ and open $G \supseteq A$ such that

$$P(G - F) < \varepsilon,$$

$\varepsilon > 0$ arbitrary.

Proof of Thm 1.1

Lets identify a class $\mathcal{G}$ of sets in S that satisfies the prop.

(i) (Put the closed sets in.)

Suppose A is closed.

Choose $F = A$, and define

$$G^{\delta} := \{ x \in S : \rho(x, A) < \delta \}.$$

Then,

$$P(G^{\delta} - F) = P(G^{\delta}) - P(F)$$

However, if $\delta \downarrow 0$, then

$$\bigcap_{\delta} G^{\delta} \downarrow \bar{F} = F \quad *$$

and hence

$$P\left(\bigcap_{\delta} G^{\delta}\right) \downarrow P(F) \text{ as } \delta \downarrow 0. \; *$$

Note: S is open and closed.

(ii) (closed under complements)

Suppose $A \subseteq S$ satisfies the assertion. Then A^c does as well.

(iii) (closed under countable additivity).

Suppose $A_n \subseteq S$ satisfies the assertion; that is, for $\varepsilon_n > 0$

$\exists$ F_n closed and G_n open so that

$$F_n \subseteq A_n \subseteq G_n, \quad n \geq 1$$

and $P(G_n - F_n) \leq \varepsilon_n$.

Choose $\varepsilon_n = \varepsilon / 2^{n+1}$.

Hence,

$$\bigcup_n F_n \subseteq \bigcup A_n \subseteq \bigcup_n G_n$$

(not nec. closed) (open)

Find n_0 so that

$$P\left(\bigcup_n F - \bigcup_{n \leq n_0} F\right) \leq \frac{\varepsilon}{2}.$$

and notice that

$$\bigcup_{n \leq n_0} F \subseteq \bigcup_n A_n \subseteq \bigcup_n G_n$$

Hence $\mathcal{G}$ is a σ-algebra containing the open sets.

Conclude $\mathcal{G} \supseteq \mathcal{B}$.

A subclass $\mathcal{S}' \subset \mathcal{S}$ is called a **separating class** if a probability measure is "completely decided" by its values on $\mathcal{S}'$, that is, if

$$P(A') = Q(A') \quad \forall \, A' \in \mathcal{S}'$$

then

$$P(A) = Q(A) \quad \forall \, A \in \mathcal{S}$$

Theorem 1.1 implies that the closed sets in $\mathcal{S}$ form a separating class, that is, they completely decide the probability measure.

Theorem 1.2

Probability measures P and Q on $\mathcal{S}$ coincide if $Pf = Qf$ $\forall f \in \mathcal{B}C(S)$.

Proof of Theorem 1.2

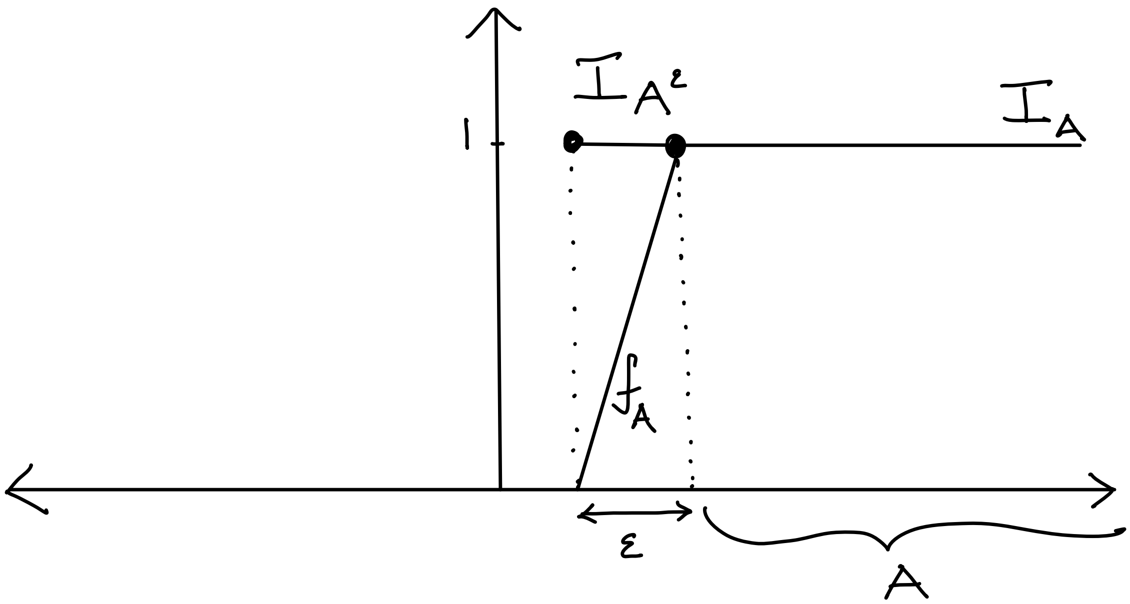
Suppose that for any closed $A \in \mathcal{S}$ we are able to find a function $f_A \in BC(S)$, such that $I_A \leq f_A \leq I_{A^\varepsilon}$ where

$$A^\varepsilon := \{x \in S : \rho(x, A) \leq \varepsilon\}.$$

Then, notice:

$$P(A) = PI_A \leq Pf_A = Qf_A \leq Q(A^\varepsilon)$$

Since $Q(A^\varepsilon) \downarrow Q(\bar{A})$ as $\varepsilon \downarrow 0$, above implies $P(A) \leq Q(A)$ if A is closed. Similarly, $Q(A) \leq P(A)$.



$$f_A(x) = \left(1 - \frac{\rho(x, A)}{\varepsilon}\right)^+$$

Notice that

$$I_A \le f_A \le I_{A^\varepsilon} .$$

A probability measure P on $(S, \mathcal{S})$ is tight if for each $\varepsilon > 0$, $\exists$ a compact set K such that

$$P(K) \geq 1 - \varepsilon.$$

A family Π of probability measures on $(S, \mathcal{S})$ is tight if each measure in Π is tight.

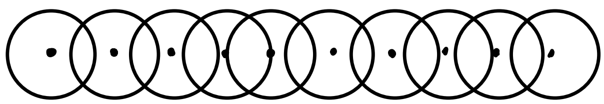
We will show later that if Π is tight, then it is relatively compact.

Theorem 1.3

If S is separable and complete, then each probability measure on $(S, \mathcal{S})$ is tight.

Proof of Thm 1.3

Complete + Totally Bounded $\Longleftrightarrow$ relatively compact.



Since S is separable we can find a countable number of balls A_{jk}, $j = 1, 2, \ldots$ each of radius $\frac{1}{k}$ that covers S.

We can find n_k so that

$$P\Big(\bigcup_{j \leq n_k} A_{jk}\Big) \geq 1 - \frac{\varepsilon}{2^k}$$

- Also, notice that since S is complete, the closure of $\bigcap_{k\geq 1} \bigcup_{j\leq n_k} A_{jk}$ is complete

- The set $\bigcap_{k\geq 1} \bigcup_{j\leq n_k} A_{jk}$ is totally bounded.

Conclude $K = \bigcap_{k\geq 1} \bigcup_{j\leq n_k} A_{jk}$ has compact closure. Also $P(K) > 1-\varepsilon$ since

$$P(K^c) = P\Big(\bigcup_{k\geq 1}\Big(\bigcup_{j\leq n_k} A_{jk}\Big)^c\Big) \leq \varepsilon.$$