Recall:

$$P_n \Longrightarrow P$$

means

$$P_n f \longrightarrow Pf \qquad \text{——— (1)}$$

for all bounded continuous real-valued functions on S.

(We will take (1) as the definition of weak convergence.)

$(S, \mathcal{S})$ is some arbitrary space, and let $x_n, x \in S$

$\boxed{\text{Example 2.1}}$

Define the sequence of measures

$$P_n \equiv \delta_{x_n} \quad ; \quad P \equiv \delta_x$$

$$\left(\delta_y(A) = \begin{cases} 1 & \text{if } y \in A \\ 0 & \text{o.w.} \end{cases} \right)$$

$$\boxed{\text{Claim: } P_n \Longrightarrow P \text{ if and only if } x_n \longrightarrow x.}$$

Let's prove.

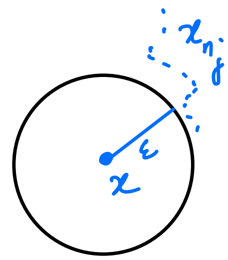
Suppose $x_n \longrightarrow x$.

Then

$$P_n f = f(x_n) \longrightarrow f(x) = Pf.$$

Now suppose $x_n \not\to x$

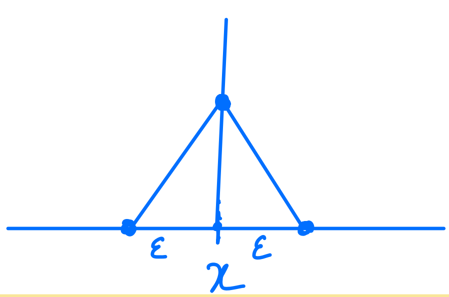
We can find $\{x_{n_j}, j \geq 1\}$



s.t. $\rho(x, x_{n_j}) > \varepsilon$.

Consider the function

$$f(y) = \left(1 - \frac{\rho(y,x)}{\varepsilon}\right)^+$$



Notice:

$$P_{n_j} f = f(x_{n_j}) = 0$$

$$\neq f(x) = Pf = 1.$$

$\blacksquare$

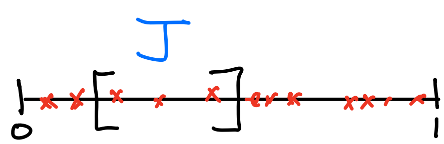
$\boxed{\text{Example 2.2}}$ (Fundamental Thm of Simulation.)

$$S = [0, 1] \; ; \; \rho(x, y) = |x - y|$$

Let $x_{n1}, x_{n2}, \ldots x_{nr_n}$ be r_n pts "sprinkled unif." on $[0, 1]$:

For any interval $J \subseteq [0, 1]$,

$$\frac{1}{r_n} \sum_{j=1}^{r_n} \mathbb{I}_J(x_{nj}) \longrightarrow |J| .$$



P_n assigns a "point mass" at each x_{nj}, $j = 1, 2, \ldots, r_n$.

$$P_n(A) = \frac{1}{r_n} \sum_{j=1}^{r_n} \mathbb{I}_A(x_{nj})$$

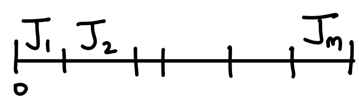
$$\boxed{\text{Claim: } P_n \Rightarrow P \text{ where } P \text{ is the Lebesgue measure on } [0,1].}$$

bounded, cont., real-valued fns on [0,].

Proof

Consider any $f \in \mathcal{B}C[0,1]$.

Since f is Riemann, $\exists$ sub-intervals J_i s.t. *



$$\int f - \sum_i l_i |J_i| \le \varepsilon$$

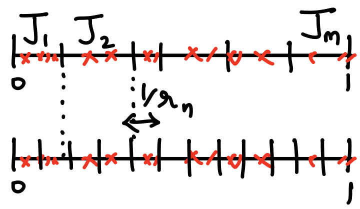
$$\sum_i u_i |J_i| - \int f \le \varepsilon$$

$$\left(l_i = \inf_{x\in[0,1]} f(x) \; ; \; u_i = \sup_{x\in[0,1]} f(x) \right)$$

Proof (contd...)

$$\sum_i l_i |J_i| \leq \int f \leq \sum_i u_i |J_i|$$

Notice, however,



$$P_n f = \frac{1}{r_n} \sum_{j=1}^{r_n} f(x_{nj}) \geq \sum_i l_i |J_i|$$

$$P_n f = \frac{1}{r_n} \sum_{j=1}^{r_n} f(x_{nj}) \leq \sum_i u_i |J_i|$$

Conclude: $P_n f \to \int f = Pf$.

Portmanteau Theorem 2.1

S, 8

The following are equivalent.

(i) $P_n \Longrightarrow P$

(ii) $P_n f \longrightarrow Pf$ for all bounded unif. cont. f.

(iii) $\limsup_n P_n F \leq PF$ for all closed F.

(iv) $\liminf_n P_n G \geq PG$ for all open G.

(v) $P_n A \longrightarrow PA$ for P-continuous sets A.

Proof Sketch

(i) $\Longrightarrow$ (ii) is trivial.

(ii) $\Longrightarrow$ (iii).

Invoke our old friend

$$f_F(x) = \left(1 - \frac{\rho(x,F)}{\varepsilon}\right)^+$$

$$\begin{aligned}\limsup_n P_n F &\le \limsup_n P_n f_F \\ &= P f_F \le P f_{F^\varepsilon} \\ &= P F^\varepsilon.\end{aligned}$$

As $\varepsilon \downarrow 0$, $PF^\varepsilon \downarrow P\bar{F} = PF$.
since F is closed. *

Proof Sketch

(iii) $\Longrightarrow$ (iv)

G^c is closed and so

$$\limsup_n P_n G^c \leq P G^c$$

$$\Rightarrow \limsup_n 1 - P_n G \leq 1 - PG$$

$$\Longrightarrow 1 - \liminf_n P_n G \leq 1 - PG$$

$$\Longrightarrow \liminf_n P_n G \geqslant PG.$$

Proof Sketch

(iii) $\implies$ (v)

A° is the interior of A

$\bar{A}$ is the closure of A

$$\left(A - \partial A = A^\circ ; \quad A + \partial A = \bar{A} \right).$$

$$\limsup_n P_n A \leq \limsup_n P_n \bar{A} \leq P\bar{A}$$

$$\liminf_n P_n A \geq \liminf_n P_n A^\circ \geq P A^\circ$$

Since $P\partial A = 0$, $\quad P\bar{A} = PA^\circ = PA.$

Conclude $P_n A \longrightarrow PA.$

Proof Sketch

(v) $\implies$ (i)

Consider $f \in \mathcal{B}C(S)$ and say WLOG that $0 < f < 1$.

$$P_n f \overset{*}{=} \int_0^\infty P_n(f>t)\,dt$$

$$= \int_0^1 P_n(f>t)\,dt \overset{?}{\longrightarrow} \int_0^1 P(f>t)\,dt = Pf.$$

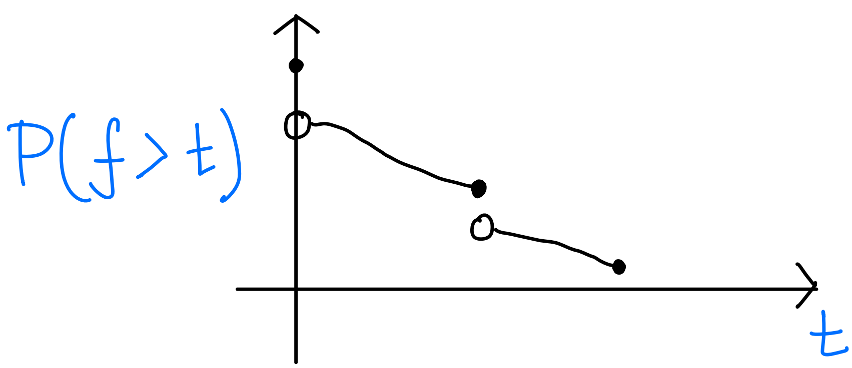
However,

$$\lim_n \int_0^1 P_n(f>t)\,dt \overset{BCT}{=} \int_0^1 \lim_n P_n(f>t)\,dt$$

Proof Sketch

$$\lim_n P_n(f > t) = P(f > t)$$

by (v) if $(f > t)$ is a P-continuity set. Also, $\partial(f > t) \subseteq (f = t)$. *



$P(f > t)$ has at most countable no. of jumps. *

Usually $\mathcal{S}$ is "too big." So, we look for a convenient sub-class of $\mathcal{S}$ which will ensure weak convergence.

Theorem 2.2

$\mathcal{A}_p \subseteq \mathcal{S}$. Suppose

(i) $\mathcal{A}_p$ is a π-system; and

(ii) $\mathcal{A}_p$ generates the open sets.

If $P_n A \to PA$ for $A \in \mathcal{A}_p$,
then $P_n \Rightarrow P$.

Theorem 2.2 Proof Sketch

Let $G \in \mathcal{S}$ be open.

— due to (ii), $\exists\, A_j \in \mathcal{A}_p$ s.t.

$$G = \bigcup_j A_j \qquad (1)$$

— due to (i), $\forall r$

$$* \quad P_n\left(\bigcup_{j=1}^{r} A_j\right) \longrightarrow P\left(\bigcup_{j=1}^{r} A_j\right) \qquad (2)$$

$\boxed{\text{Theorem 2.2 Proof Sketch}}$

contd...

Choose r so that

$$* \quad P_n\left(\bigcup_{j=1}^{r} A_j\right) > P(G) - \varepsilon. \tag{3}$$

Then,

$$\liminf P_n(G) \overset{(1)}{\geq} \liminf P_n\left(\bigcup_{j=1}^{r} A_j\right)$$

$$\xrightarrow{(2)} P\left(\bigcup_{j=1}^{r} A_j\right) \overset{(3)}{>} P(G) - \varepsilon.$$

Theorem 2.2 is useful but checking (ii) can be inconvenient.

Theorem 2.3

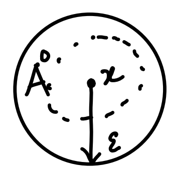
$\mathcal{A}_P \subseteq \mathcal{S}$. Suppose

(i) $\mathcal{A}_P$ is a π-system;

(ii) S is separable; and

(iii) for each $x \in S$ and $\varepsilon > 0$
$\exists\, A \in \mathcal{A}_P$ so that

$$x \in A^\circ \subseteq A \subset B(x, \varepsilon).$$



If $P_n A \to PA$ for $A \in \mathcal{A}_P$
then $P_n \Rightarrow P$.

Theorem 2.3 Proof Sketch

Take any $G \in \mathcal{S}$ open.

Due to (ii) & (iii), we can find $A_j \in \mathcal{A}_p$ such that

$$G = \bigcup_j A_j^\circ .$$

Now follow proof of Thm. 2.2.

Notice that Thms. 2.1 — 2.3 essentially characterize a class smaller than $\mathcal{S}$ which guarantee weak convergence.

Lets make this formal . . .

A subclass $\mathcal{A}$ of $\mathcal{S}$ is called a convergence determining class if

$$P_n A \longrightarrow PA \quad \forall A \in \mathcal{A}$$

implies

$$P_n \Longrightarrow P.$$

$\left(\begin{array}{l} \text{Convergence det.} \Longrightarrow \text{Separating} \\ \text{Separating} \not\Longrightarrow \text{Convergence det.} \end{array}\right)$

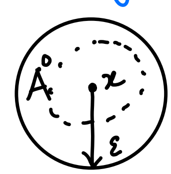
Theorem 2.4

$\mathcal{A} \subseteq \mathcal{S}$. Suppose

(i) $\mathcal{A}$ is a π-system; and

(ii) S is separable.

$\mathcal{A}_{x,\varepsilon}$:= class of sets satisfying (iii) of Thm. 2.3.



(iii) $\partial \mathcal{A}_{x,\varepsilon}$ contains $\emptyset$ or uncountably many disjoint sets.

Then $\mathcal{A}$ is a <u>conv. det. class</u>.

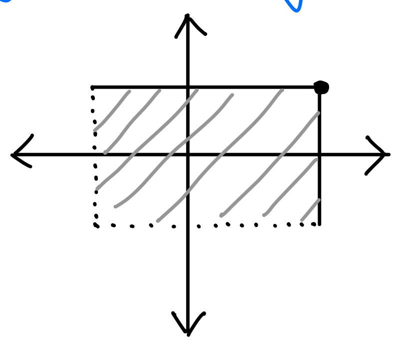
Let's look at examples of conv. det. classes . . .

Example I

$\mathcal{R}^k$:= k-dimensional Borel σ-algebra

The class $\mathcal{A}$ containing rectangles

$$\left\{ y : a_i < y_i \leq b_i, \; i \leq k \right\}$$

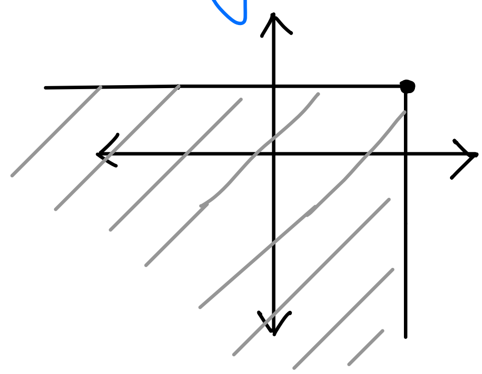


is a conv. determining class.

Why? It satisfies postulates of Thm. 2.4.

Example I contd ...

The class $\mathcal{A}$ containing "southwest quadrants"



$$Q_x := \left\{ y : y_i \leq x_i ,\ i \leq k \right\}$$

is a convergence determining class.

Example II

$$S = \mathbb{R}^{\infty} := \{x = (x_1, x_2, \dots), x_j \in \mathbb{R}\}$$

$$\pi_k(x) = (x_1, x_2, \dots, x_k), \; x \in \mathbb{R}^{\infty}.$$

For $H \in \mathcal{R}^k$

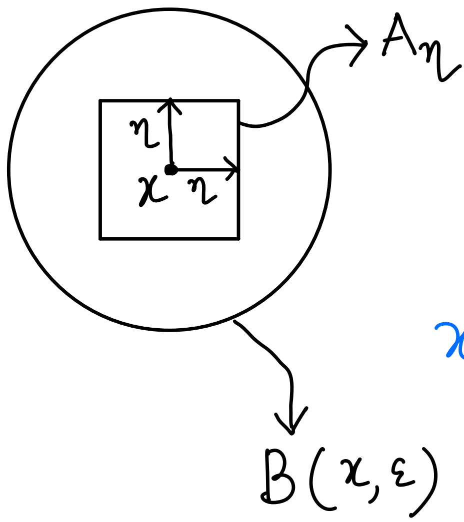
$$\pi_k^{-1}(H) := \{x \in \mathbb{R}^{\infty} : \pi_k(x) \in H\}.$$

$$\mathcal{R}_f^{\infty} := \{\pi_k^{-1}(H), \; H \in \mathcal{R}^k, \; k \geq 1\}$$

(H is the "k-dim. Borel set")

Example II contd...

Consider $x \in \mathbb{R}^\infty$, $\varepsilon > 0$.



Choose $2^{-k} < \varepsilon/2$; $0 < \eta < \frac{\varepsilon}{2}$

$$x \in A_\eta^\circ \subset A_\eta \subset B(x,\varepsilon).$$

$$A_\eta := \{ y : |y_i - x_i| \le \eta \text{ for } i \le k, k \ge 1 \}.$$

$$\partial A_\eta := \{ y \in A_\eta : |y_i - x_i| = \eta, \text{ some } i \}.$$

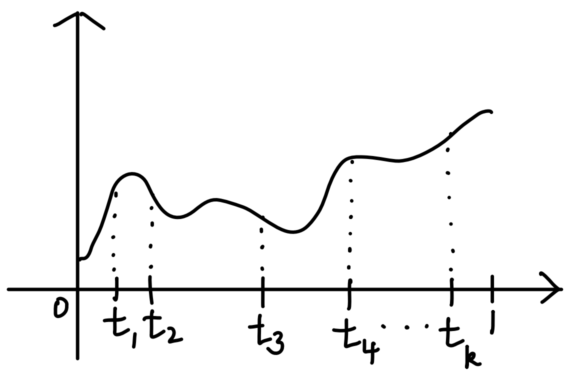
Hence boundaries are disjoint, Thm. 2.4 applies

$\mathbb{R}_f^\infty$ is a conv. det. class.

Example III

$S = C[0,1]$ is the space of continuous functions with

$$\rho(x,y) = \sup_t |x(t) - y(t)|$$

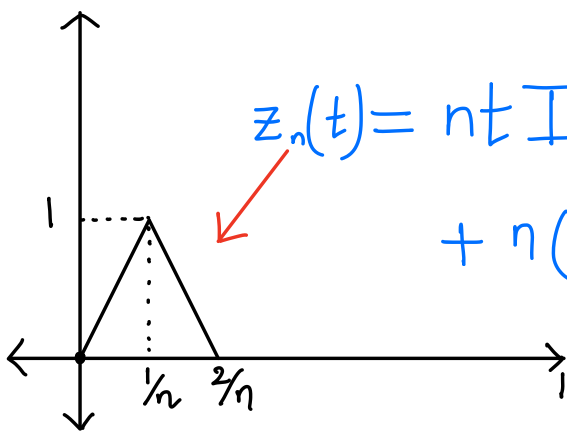


$$\Pi_{t_1 \cdots t_k}(x) = \left(x(t_1), \ldots x(t_k)\right)$$

$$C_f := \left\{ \Pi^{-1}_{t_1 \cdots t_k}(H),\ H \in \mathbb{R}^k,\ k \geq 1 \right\}$$

Example III (contd...)

Consider the following.



$$z_n(t) = nt\, I_{[0,1/n]}(t) + n\left(\frac{2}{n} - t\right) I_{(1/n,\,2/n]}(t), \quad t \in [0,1]$$

$$z(t) = 0, \quad t \in [0,1]$$

$$P_n := \delta_n, \quad P := \delta.$$

$P_n \not\Rightarrow P$ because $\rho(z_n, z) = 1 \not\to 0.$

Example III (contd...)

However, consider any $A \in \mathcal{C}_f$.

$$A = \pi^{-1}_{t_1 \dots t_k}(H) \text{ for some } k$$
$$\text{and } H \in \mathcal{R}^k.$$

$$PA = \begin{cases} 1 & \text{if } A \ni 0 \\ 0 & \text{if } A \not\ni 0 \end{cases}$$

And,

$$P_n(A) = 0 \quad \text{if } \frac{2}{n} < t_1.$$

So, we have found P_n, P so that

$P_n \not\Rightarrow P$ but $P_n(A) \longrightarrow P(A)$, $A \in \mathcal{C}_f$

$\mathcal{C}_f$ is not a conv. det. class.

Some more terminology.

some metric space

— for $h : S \longrightarrow S'$, we say h is measurable $\mathcal{S}/\mathcal{S}'$ if $h^{-1}(A) \in \mathcal{S}$ for $A \in \mathcal{S}'$.

— the prob. measure Ph^{-1}

$$Ph^{-1}(A) := P(h^{-1}(A))$$

We need conditions under which $P_n \Longrightarrow P$ implies $P_n h^{-1} \Longrightarrow P h^{-1}$.

$h : S \to S'$ continuous will do the job.

$\boxed{\text{Theorem } 2.7}$ (Mapping Thm)

Suppose h is measurable $\mathcal{S}/\mathcal{S}'$, and $D_h \in \mathcal{S}$ is the set of discontinuities of h.

If $P_n \Longrightarrow P$ and $P(D_h) = 0$, then $P_n h^{-1} \Longrightarrow P h^{-1}$.

$\boxed{\text{Theorem 2.7 Proof Sketch}}$

Take a closed set $F \in \mathcal{S}'$.

We are done if we can prove

$$\limsup_n P_n h^{-1}(F) \leq P h^{-1}(F).$$

(due to Portmateau).

<u>Notice</u>:

$$P_n h^{-1}(F) := P_n\left(h^{-1}(F)\right)$$

$$\leq P_n\left(\overline{h^{-1}(F)}\right)$$

Theorem 2.7 Proof Sketch

If h is continuous at $x \in S$ and $x \in \overline{h^{-1}(F)}$, then $x \in h^{-1}(\overline{F})$.

(*)

Therefore,

$$\overline{h^{-1}(F)} \cap D_h^c \subseteq h^{-1}(\overline{F})$$

and

$$
\begin{aligned}
P_n\left(\overline{h^{-1}(F)}\right) &= P_n\left(\overline{h^{-1}(F)} \cap D_h^c\right) \\
&\longrightarrow P\left(\overline{h^{-1}(F)} \cap D_h^c\right) \\
&\leq P\left(h^{-1}(\overline{F})\right) = P\left(h^{-1}(F)\right)
\end{aligned}
$$

Example Application

(Random Variate Generation)

Let $F : \mathbb{R} \longrightarrow [0,1]$ be a distbn function.

$$\varphi(u) := \inf \{ x : F(x) \geq u \}$$

quantile function.

$\varphi : [0,1] \longrightarrow \mathbb{R}$ is non-decreasing and hence D_φ is countable.

Example Application (contd)

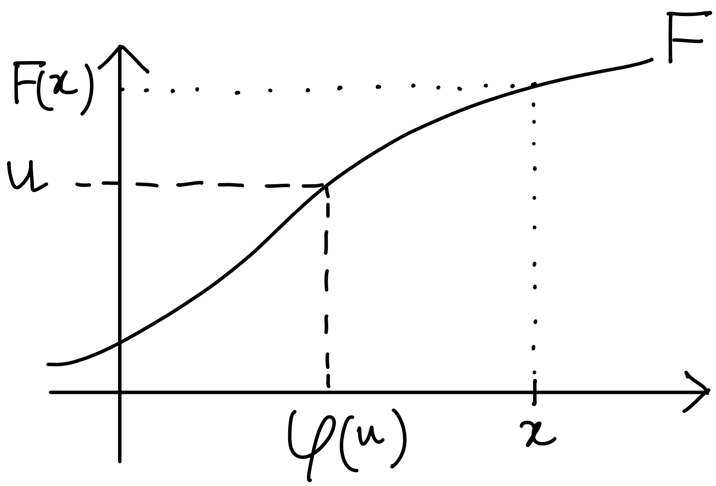
(Random Variate Generation)

Suppose P is the Lebesgue meas. on $[0, 1]$. Then $P\varphi^{-1}$ has the distribution function F.

Lets see why.

Example Application (contd)

(Random Variate Generation)



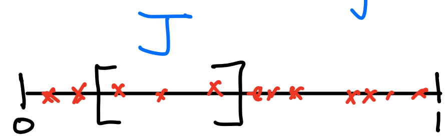
$$P\varphi^{-1}(-\infty, x] = P\big(u : \varphi(u) \le x\big)$$

$$= P\big(u : F(x) \geq u\big)$$

$$= F(x).$$

Example Application (contd)
(Random Variate Generation)

Let P_n be the measure from Example 2.2.



Recall that $P_n \Longrightarrow P$.

Since $PD_\varphi = 0$, conclude from the mapping theorem that

$$P_n \varphi^{-1} \Longrightarrow P\varphi^{-1}$$