Random Elements

S-valued random variable.

Suppose $(\Omega, \mathcal{F}, P)$ is probability space, and let

$X : \Omega \longrightarrow S$ be measurable

$\mathcal{F}/\mathcal{S}$, where S is a metric space.

random variable random vector random sequence random function	X on Ω X in S

The distribution of X is the measure $Q := \underbrace{P}_{P}X^{-1}$ on $(S, \mathcal{S})$ defined as

$$Q(A) := PX^{-1}(A)$$

$$= P(\omega : X(\omega) \in A).$$

$$=: P(X \in A).$$

P is defined on an arbitrary $\mathcal{F}$.

Q is defined on the Borel $\sigma(S)$.

Notice

I. Suppose f is measurable $\mathcal{S}/\mathcal{R}^1$, then

$$\mathbb{E}[f(X)] = \int f(X(\omega))\, dP(\omega)$$

$$= \int f(x)\, dQ(x)$$

$$\int f(X(X^{-1}(x)))\, dP(X^{-1}(x))$$

$$= Qf.$$

II. Each prob. measure on each metric space is the distribution of some random variable on some probability space.

$$(\Omega, \mathcal{F}, P) \qquad (S, \mathcal{S}, Q)$$

$X \perp$

<u>Notice</u>

III. Suppose X_n has distbn. P_n and X has distbn. P.

We say $X_n \Longrightarrow X$

("conv. in distbn." or "conv. weakly")

if $P_n \Longrightarrow P$.

The underlying prob. spaces almost never enter the fray.

Portmanteau Theorem

(i) $X_n \Longrightarrow X$

(ii) $\mathbb{E}[f(X_n)] \longrightarrow \mathbb{E}[f(X)]$
for bounded unif. cont. f.

(iii) for closed F,
$\limsup_n P(X_n \in F) \leq P(X \in F)$ (*)

(iv) for open G, . . .

(v) $P(X_n \in A) \longrightarrow P(X \in A)$, $A \in \partial X$.

Just symbols!

$$P_n \Longrightarrow P$$

$$X_n \Longrightarrow X$$

$$X_n \Longrightarrow P$$

$$P_n \Longrightarrow X$$

For an <u>element</u> $a \in S$,

We say $X_n \xrightarrow{P} a$ if

"conv. in prob."

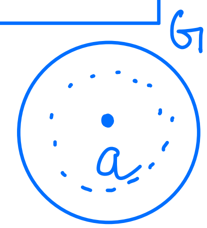
$$P\left(\rho(X_n, a) > \varepsilon\right) \to 0 \quad \forall \varepsilon > 0.$$

- "a" is a const-valued r.v.
- For the above, X_n, $n \geq 1$ all live in the same space Ω.

$$\boxed{\text{Theorem } 3.0}$$

$X_n \Longrightarrow \delta_a$ iff $X_n \xrightarrow{P} a$.

Theorem 3.0 Proof Sketch



Suppose $X_n \xrightarrow{P} a$.

Take open G and suppose $a \in G$.

$$\liminf_n P_n(G) \geq \liminf_n P(\rho(X_n, a) \leq \varepsilon) = 1.$$

(If $a \notin G$ $P_n(G) \geq 0 = P(a \in G)$)

Now suppose $X_n \Longrightarrow a$. Then

$$P(\rho(X_n, a) > \varepsilon) =: P_n(B^c(a, \varepsilon)) \longrightarrow P(B^c(a, \varepsilon)) = 0.$$

$\boxed{\text{Theorem 3.1}}$ "Slutsky"

Suppose (X_n, Y_n) are <u>random elements</u> of $S \times S$. If

$$X_n \Longrightarrow X \quad \& \quad \rho(X_n, Y_n) \xrightarrow{P} 0$$

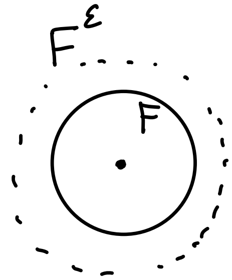
then

$$Y_n \Longrightarrow X$$

(<u>Slutsky's Thm</u>: $X_n \Longrightarrow X$ and $A_n \xrightarrow{P} A$, $B_n \xrightarrow{P} B$, then $A_n X_n + B_n \Longrightarrow AX + B$) (*)

Theorem 3.1 Proof Sketch

Take F closed in $\mathcal{S}$.



$$F^{\varepsilon} := \{x : \rho(x, F) \le \varepsilon\}$$

<u>Notice:</u>

$$(Y_n \in F) \subseteq (X_n \in F^{\varepsilon}) \cup (\rho(X_n, Y_n) > \varepsilon)$$

Therefore,

$$P(Y_n \in F) \le P(X_n \in F^{\varepsilon}) + P(\rho(X_n, Y_n) > \varepsilon)$$

Take lim sup and apply Portmanteau, and then $\varepsilon \downarrow 0$.

Theorem 3.2

Suppose (X_{un}, X_n) are random elements of $S \times S$. If

$$X_{un} \Rightarrow_n Z_u \Rightarrow_u X$$

and

$$\lim_u \limsup_n P\left(\rho(X_{un}, X_n) \geq \varepsilon\right) = 0$$

for $\varepsilon > 0$, then $X_n \Rightarrow_n X$.

Theorem 3.2 Proof Sketch

Following the proof of Thm. 3.1,

$$P(X_n \in F) \leq P(X_{un} \in F^\varepsilon) + P(\rho(X_{un}, X_n) > \varepsilon)$$

$$\limsup_n P(X_n \in F) \leq P(Z_u \in F^\varepsilon) + \limsup_n P(\rho(X_{un}, X_n) > \varepsilon) \xrightarrow[u]{} P(X \in F^\varepsilon).$$

Now send $\varepsilon \downarrow 0$.

Integration to the Limit

If $X_n \Longrightarrow X$, what can we say about $\mathbb{E}[X_n]$ in relation $\mathbb{E}[X]$?

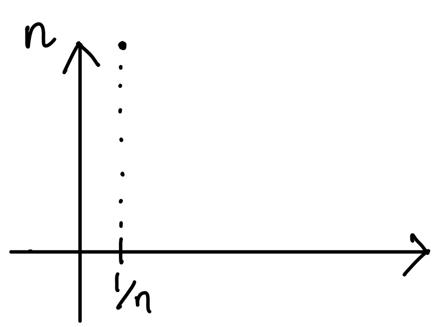
Theorem 3.4 (Fatou's Lemma)

For <u>random variables</u> X_n, X

if $X_n \Longrightarrow X$, then

$$\mathbb{E}[|X|] \le \liminf_n \mathbb{E}[|X_n|].$$

To get the inequality direction:



$$X_n = \begin{cases} n & \text{wp } 1/n \\ 0 & \text{ow} \end{cases}$$

$$X = 0 \quad \text{wp } 1$$

$\boxed{\text{Theorem 3.4 Proof Sketch}}$

Due to mapping thm,

$$P(|X_n| > t) \to P(|X| > t). \quad \text{—— (1)}$$

$$\mathbb{E}[|X|] \overset{(*)}{=} \int P(|X| > t)$$

$$\overset{(1)}{=} \int \lim_n P(|X_n| > t)$$

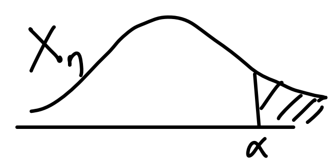
$$\overset{\text{Fatou}(*)}{\leq} \liminf_n \int P(|X_n| > t).$$

Theorem 16.3. *For nonnegative f_n,*

$$\int \liminf_n f_n \, d\mu \leq \liminf_n \int f_n \, d\mu. \tag{16.6}$$

The <u>random variable</u> sequence $\{X_n, n \geq 1\}$ is <u>uniformly integrable</u> (U.I) if

$$\lim_{\alpha \to \infty} \sup_{n} \int_{|X_n| \geq \alpha} |X_n| \, dP = 0.$$



If $\{X_n, n \geq 1\}$ is U.I, then

$$\mathbb{E}\left[|X_n|\right] \leq 1+\alpha \quad \forall n.$$

$\boxed{\text{Theorem 3.5}}$

If $\{X_n, n \geq 1\}$ are U.I

and $X_n \Longrightarrow X$, then

X is integrable (*) and

$$\mathbb{E}[X_n] \to \mathbb{E}[X].$$

(X is <u>integrable</u> if $\mathbb{E}[|X|] < \infty$

$\Longleftrightarrow \mathbb{E}[X^+] < \infty$ & $\mathbb{E}[X^-] < \infty$.)

Theorem 3.5 Proof Sketch

$\mathbb{E}[|X_n|] \leq 1+\alpha \quad \forall n.$

Hence, $\mathbb{E}[|X|] < \infty$. and

<u>X is integrable</u>.

Also, it is okay to prove

$\mathbb{E}[X_n] \longrightarrow \mathbb{E}[X]$ assuming

X_n, X are non-negative. <u>Why</u>?

$\boxed{\text{Theorem 3.5 Proof Sketch}}$

$$\mathbb{E}[X_n] = \int X_n \mathbb{I}(X_n < \alpha)\, dP + \int_{X_n \geq \alpha} X_n \, dP$$

$$= \int_0^\alpha P(t < X_n < \alpha)\, dt + \int_{X_n \geq \alpha} X_n \, dP$$

Similarly, $\leq \varepsilon$ for large α

$$\mathbb{E}[X] = \int_0^\alpha P(t < X < \alpha)\, dt + \int_{X \geq \alpha} X \, dP$$

Theorem 3.5 Proof Sketch

And, there are at most countable number of t, α such that

$$P(t < X_n < \alpha) \not\to P(t < X < \alpha)$$

Use bounded convergence thm (*) to conclude.

Theorem 16.5. *If $\mu(\Omega) < \infty$ and the f_n are uniformly bounded, then $f_n \to f$ almost everywhere implies $\int f_n \, d\mu \to \int f \, d\mu$.*

Useful U.I Condition.

If $\sup_n \mathbb{E}\left[|X_n|^{1+\varepsilon}\right] < \infty$, some $\varepsilon > 0$

then $\{X_n, \ n \geq 1\}$ is U.I.

Notice

$$\mathbb{E}\left[|X_n|^{1+\varepsilon}\right] = \int_{|X_n| \geq \alpha} |X_n|^{1+\varepsilon} dP + \int_{|X_n| < \alpha} |X_n|^{1+\varepsilon} dP$$

$$\geq \int_{|X_n| \geq \alpha} |X_n| \alpha^{\varepsilon} dP$$

Therefore,

$$\int_{|X_n| \geq \alpha} |X_n| dP \leq \frac{\mathbb{E}\left[|X_n|^{1+\varepsilon}\right]}{\alpha^{\varepsilon}}.$$