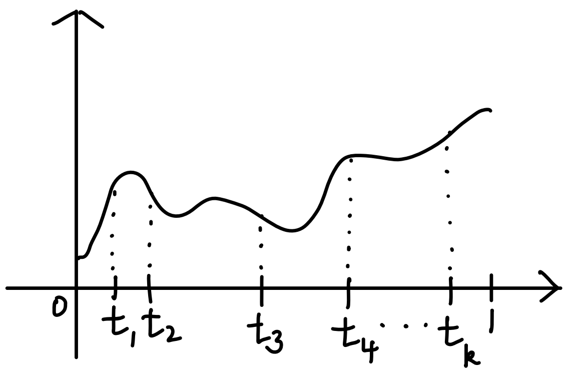
Let's go back to an important negative example.

Consider:



$S = C[0,1]$

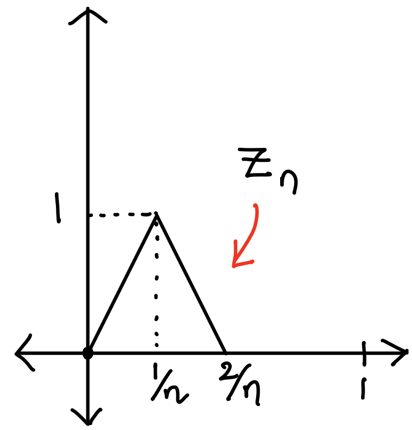
ρ is the sup-norm

$$\Pi_{t_1 \dots t_k}(x) = \left(x(t_1), \dots x(t_k)\right)$$

"projection r.v."

$$C_f := \left\{ \Pi^{-1}_{t_1 \dots t_k}(H),\ H \in \mathbb{R}^k,\ k \geq 1 \right\}$$

$P_n = \delta_{z_n}$; $P = \delta_0$



We showed that

$$P_n(A) \longrightarrow P(A) \quad \forall A \in \mathcal{C}_f$$

but

$$P_n \not\Rightarrow P.$$

Finite-dimensional distributions converge but no weak convergence!

Why? What further condition(s) ensure that convergence of finite-dim. distbns. imply weak conv.?

$\boxed{\text{Answer: Relative Compactness.}}$

$\boxed{\text{Theorem. (Appendix M)}}$

The following are equivalent.

(i) $\bar{A}$ is compact.

(ii) Each sequence in A has a convergent subsequence. (with limit in $\bar{A}$).

(iii) A is totally bounded and $\bar{A}$ is complete.

A family Π of probability measures on $(S, \mathcal{S})$ is <u>relatively</u> <u>compact</u> if each sequence $\{P_n, n \geq 1\}$ has a weakly convergent subsequence. $\{P_{n_j}, j \geq 1\}$, that is, $\exists\ Q$ such that $P_{n_j} \Rightarrow Q$.

Similarly, $\{P_n, n \geq 1\}$ is <u>relatively compact</u> if every sub-sequence $\{P_{n_j}, j \geq 1\}$ has a weakly convergent sub-subsequence:

$\exists\ Q$, $\{P_{n_{j(m)}}, m \geq 1\}$ such that

$$P_{n_{j(m)}} \Longrightarrow Q.$$

$\boxed{\text{Theorem 5.1}}$ (Fin. Dim. Conv + RC $\Rightarrow$ Weak. Conv.)

Suppose probability measures P_n, P on $(C, \mathcal{C})$ are such that

$$P_n \pi^{-1}_{t_1 t_2 \cdots t_k} \Rightarrow P \pi^{-1}_{t_1 t_2 \cdots t_k}. \quad \text{—— (1)}$$

If $\{P_n, n \geq 1\}$ is RC, then

$$P_n \Rightarrow P.$$

Theorem 5.1 Proof Sketch

Since P_n is RC, $\exists\, Q$, $P_{n_{j(m)}}$

s.t. $P_{n_{j(m)}} \Rightarrow_m Q$. —— (2)

and hence (by mapping thm)

$$P_{n_{j(m)}} \pi^{-1}_{t_1 t_2 \cdots t_k} \Rightarrow Q \pi^{-1}_{t_1 t_2 \cdots t_k}.$$

$\forall k, t_1, t_2, \ldots t_k$.

—— (3)

(1) and (3), and since $\mathcal{C}_f$ is a separating class, we have $P = Q$.

$\boxed{\text{Theorem 5.1 Proof Sketch}}$

contd...

Therefore, from (1), every sub-sequence of $\{P_n, n \geq 1\}$ has a sub-sub sequence converging weakly to P.

(*) Hence,

$$P_n \Longrightarrow P.$$

▨

Theorem 5.1 does not assume too much because $P_n \Longrightarrow P$ implies $\{P_n, n \geq 1\}$ is RC.

Notice that Thm. 5.1 postulates the existence of a measure P on $(C, \mathcal{C})$. What if we only know that

$$P_n \pi^{-1}_{t_1 \ldots t_k} \Rightarrow_n \mu_{t_1 \ldots t_k}$$

on $(\mathbb{R}^k, \mathcal{R}^k)$?

Theorem 5.2 (Existence)

Suppose $\{P_n, n \geq 1\}$ on $(C, \mathcal{C})$ is RC and

$$P_n \pi^{-1}_{t_1 t_2 \cdots t_k} \Rightarrow_n \mu_{t_1 t_2 \cdots t_k}$$

where $\mu_{t_1 \cdots t_k}$ is a probability measure on $(\mathbb{R}^k, \mathcal{R}^k)$.

Then, $\exists$ P on $(C, \mathcal{C})$ s.t.

$$P \pi^{-1}_{t_1 t_2 \cdots t_k} \Rightarrow \mu_{t_1 t_2 \cdots t_k}.$$

Theorem 5.2 Proof Sketch

Since $\{P_n, n \geq 1\}$ is RC,

$$P_{n_{j(m)}} \pi^{-1}_{t_1 \cdots t_k} \Longrightarrow_m P \pi^{-1}_{t_1 \cdots t_k}$$

$\forall\ k, t_1, \ldots, t_k$.

Therefore $P \pi^{-1}_{t_1 \cdots t_k} = \mu_{t_1 t_2 \cdots t_k}$

for all $t_1, \ldots t_k$.

$\blacksquare$

Theorems 5.1 & 5.2

are very powerful.

However, showing RC can

be difficult.

Another example.

$$\mu_n := \text{unif}[-n, n]$$

$$P_n := \begin{cases} \delta_0 & \text{even } n \\ \frac{1}{3}\delta_0 + \frac{2}{3}\mu_n & \text{odd } n \end{cases}$$

Notice that $\{P_n, n \geq 1\}$ <u>is not RC.</u>

The limit of P_n has distbn

$$F(x) = \begin{cases} 1/3 & x < 0 \\ 2/3 & x \geq 0 \end{cases}$$

Mass Escape!

Tightness is the answer.

A family Π of probability measures is tight if for every $\varepsilon > 0$, $\exists$ a compact set such that.

$$P(K) > 1 - \varepsilon.$$

(No mass escape!)

Prohorov's Thm

(a) If Π is tight, then Π is RC.

(b) If S is separable & complete, then Π is RC implies Π is tight.

$\boxed{\text{Prohorov's Thm Proof Sketch}}$

We will not prove (a).

Let's prove (b).

Consider open $G_n \uparrow S$. For each $\varepsilon > 0$, $\exists n$ s.t.

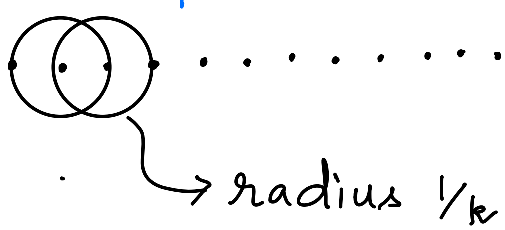
$$P G_n > 1 - \varepsilon$$

$\forall P \in \Pi$. (*)

Prohorov's Thm Proof Sketch

contd...

Since S is separable we can find open balls $A_{k1}, A_{k2}, \dots$



of radius $1/k$ covering S.

From before, $\exists\, n_k$ s.t.

$$P\left(\bigcup_{i \le n_k} A_{ki}\right) > 1 - \varepsilon/2^k$$

$\forall\, P \in \Pi$.

Prohorov's Thm Proof Sketch

contd...

Lets now construct the compact set:

$$K = \overline{\bigcap_{k \geq 1} \bigcup_{i \leq n_k} A_{ki}}$$

K is totally bounded and complete, and $P(K) > 1 - \varepsilon$.

$\blacksquare$