THE SPACE C

OVERVIEW:

ULTIMATE AGENDA: PROVE DONSKER'S THEOREM.

RECALL CENTRAL LIMIT THEOREM.

Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of iid random variables with $E\xi_1 = 0$ & $E|\xi_1|^2 = \sigma^2 < \infty$.

Then,

$$\frac{S_n}{\sigma\sqrt{n}} := \frac{\xi_1 + \xi_2 + \cdots + \xi_n}{\sigma\sqrt{n}} \Rightarrow_n \mathcal{N}(0,1).$$

Donsker's Theorem extends this result (roughly speaking) to random walks. Specifically, consider the "interpolated" random walk path,

$$X^n_t = \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) \frac{X_{\lfloor nt \rfloor + 1}}{\sigma\sqrt{n}}$$

where $t \in [0,1]$. Let $C \equiv C[0,1]$ be the space of real-valued continuous functions with domain $[0,1]$. Clearly, the interpolated path $X^n \in C$. Then, Donsker's Theorem shows that,

$$X^n \Rightarrow_n W,$$

where W is a C-valued random variable. Specifically, W has the so-called Wiener measure as its "distribution" over C.

We will leverage our understanding of weak convergence over metric spaces to prove this result.

WEAK CONVERGENCE IN C

Equip C with the uniform topology defined by the uniform metric, $\forall\, x, y \in C$,

$$\rho(x,y) := \sup_{t\in[0,1]} |x(t) - y(t)|.$$

Recall that weak convergence in C does not follow from weak convergence of finite dimensional distributions (fdd's), but also requires relative compactness.

By Prokhorov, compactness $\Longleftrightarrow$ tightness on complete & separable metric spaces. So, we have:

THEOREM 7.1 Let $\{P_n\}$, P be probability measures on $(C, \mathcal{C})$. If the fdds of $\{P_n\}$ converge to those of P & if $\{P_n\}$ is tight, then $P_n \Rightarrow_n P$.

We require a characterization of tightness on $(C, \mathcal{C})$. Note, relative compactness is not a good criterion to verify. We appeal to the Arzela - Ascoli (~~AA~~) theorem:

THEOREM 7.2 The set $A \subset C$ is relatively compact iff

(i) $\sup_{x \in A} |x(0)| < +\infty$

(ii) $\lim_{\delta \to 0} \sup_{x \in A} w_x(\delta) = 0$,

where, $w_x(\delta) := \sup_{|s-t| \leq \delta} |x(s) - x(t)|$, $0 < \delta \leq 1$.

Note: 1. $W_x(\delta)$ is the so-called modulus of continuity.

2. AA says a set of continuous functions is rel. comp. iff the functions are

(i) bounded, and

(ii) uniformly equicontinuous.

THEOREM 7.3 A Sequence of measures $\{P_n\}$ on $(C, \mathcal{C})$ are tight iff

(i) $\forall\ \eta > 0$, $\exists$ an $a \in \mathbb{R}$ & $n_0 \in \mathbb{N}$ s.t.,

$$P_n\left(x : |x(0)| \geq a \right) \leq \eta \quad , \quad \forall\ n \geq n_0.$$

(ii) $\forall\ \epsilon > 0, \eta > 0$, $\exists\ 0 < \delta < 1$ & $n_0 \in \mathbb{N}$ s.t.

$$P_n\left(x : w_x(\delta) \geq \varepsilon \right) \leq \eta \quad \forall\ n \geq n_0.$$

Proof:

WEAK CONVERGENCE OF C-VALUED RANDOM VARIABLES.

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Let $X : \Omega \to C$ be a C-valued R.V.

Let $(X_{t_1}, \dots, X_{t_k}) = \pi_{t_1,\dots,t_k}(X)$ represent the k-dimensional projection of X onto $(t_1, \dots, t_k) \in [0,1]^k$.

Let $X, X^1, X^2, \ldots$ be a sequence of R.V.'s.

THEOREM 7.5

If $(X^n_{t_1}, \ldots, X^n_{t_k}) \Rightarrow_n (X_{t_1}, \ldots, X_{t_k})$

$\forall (t_1, \ldots, t_k) \in [0,1]^k$ & $\forall k \geq 1$, and if

$$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} P\left(w(X^n, \delta) \geq \varepsilon \right) = 0 \quad \forall \varepsilon > 0,$$

then

$$X^n \Rightarrow_n X.$$

Proof:

Theorem's 7.3 & 7.5 set us up to prove Donsker's Theorem:

THEOREM 8.2

If $\xi_1, \xi_2, \ldots$ are iid $\mathbb{E}\xi_1 = 0$, $\mathbb{E}|\xi_1|^2 < +\infty$.

Define,

$$X_t^n = \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) \frac{X_{\lfloor nt \rfloor + 1}}{\sigma\sqrt{n}} \quad \forall t \in [0,1]$$

where $S_n := \sum_{i=1}^{n} \xi_i$ & $S_0 := 0$.

Then, $X^n \Rightarrow_n W$, where W is C-valued Gaussian R.V. that induces the Wiener meas.

<u>LEMMA</u> Suppose $\{\xi_n\}$ is a stationary sequence

& $$\lim_{\lambda \to \infty} \overline{\lim_{n \to \infty}} \lambda^2 P\left(\max_{k \leq n} |S_k| \geq \lambda \sigma \sqrt{n} \right) = 0.$$

Then $\{X^n\}$ is tight.

<u>Proof of Th. 8.2:</u> Verify Th. 7.5, assuming the existence of the Wiener measure.

Observe : $$(X^n_s, X^n_t - X^n_s) \Rightarrow_n (W_s, W_t - W_s)$$

by the Multivariate CLT.

The map $(x, y-x) \longmapsto (x, y)$

is continuous on $\mathbb{R}^2$, so by the continuous mapping theorem, it follows that,

$$(X^n_s, X^n_t) \Rightarrow_n (W_s, W_t),$$

for any s&t. This can be easily extended to any f.d.d.

To prove tightness we verify the Lemma:

LEMMA Suppose $\{\xi_n\}$ is a stationary sequence

& $\lim_{\lambda \to \infty} \overline{\lim_{n \to \infty}} \lambda^2 P\left(\max_{k \le n} |S_k| \geq \lambda \sigma \sqrt{n} \right) = 0$.

Then $\{X^n\}$ is tight.

Useful result : Etemadi's inequality

$$P\left(\max_{u \le m} |S_u| \geq \alpha \right) \leq 3 \max_{u \le m} P\left(|S_u| \geq \frac{\alpha}{3} \right).$$

Then it suffices to show that,

$$\lim_{\lambda \to \infty} \overline{\lim_{n \to \infty}} \lambda^2 \max_{k \le n} P\left(|S_k| \geq \lambda \sigma \sqrt{n} \right) = 0.$$

Fix λ. Observe,

$$P\left(|S_k| \geq \lambda \sigma \sqrt{n} \right) \leq P\left(|S_k| \geq \lambda \sigma \sqrt{k} \right).$$

By the central limit theorem, if $k_\lambda \leq k \leq n$,

$$P\left(|S_k| \geq \lambda \sigma \sqrt{k} \right) \leq P\left(|N| \geq \lambda \right)$$

By Markov's inequality,

$$P(|N| \geq \lambda) \leq \frac{E N^4}{\lambda^4}$$

$$= \frac{3}{\lambda^4}.$$

$$\Rightarrow \quad P(|S_k| \geq \lambda \sigma \sqrt{n}) \leq 3 \lambda^{-4}.$$

OTOH, if $k \leq k_\lambda$, by Chebyshev's inequality

$$P(|S_k| \geq \lambda \sigma \sqrt{n}) \leq \frac{Var(S_k)}{\lambda^2 \sigma^2 n}$$

$$\leq \frac{k_\lambda}{\lambda^2 n}.$$

Thus,

$$\max_{k \le n} P\left(|S_k| > \lambda \kappa \sqrt{n} \right)$$

$$\le \max_{k \le n} \left\{ \frac{3}{\lambda^4} \vee \frac{k\lambda}{\lambda^n n} \right\}.$$

$$\Rightarrow \lim_{\lambda \to \infty} \overline{\lim_{n \to \infty}} \max_{k \le n} P\left(|S_k| > \lambda \kappa \sqrt{n} \right) = 0.$$

By the Lemma, it follows $\{X^n\}$ is tight.

Since F.D.D.'s converge, it follows from Theorem 7.5 that $X^n \Rightarrow_n W$.

QED.