WIENER MEASURE: PROPERTIES & EXISTENCE.

Probability measure W on $(C, \mathcal{C})$ is called

A WIENER MEASURE IF:

1. GAUSSIAN MARGINALS

   FOR EACH $t \in [0,1]$ & $x \in C[0,1]$, $\mathcal{C}$

$$W(x_t \leq \alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{u^2}{2t}} \, du$$

2. INDEPENDENT INCREMENTS

   FOR ANY PARTITION $0 \leq t_0 \leq t_1 \leq \cdots \leq t_k = 1$,

$$x_{t_1} - x_{t_0} \perp x_{t_2} - x_{t_1} \perp \cdots \perp x_{t_k} - x_{t_{k-1}}$$

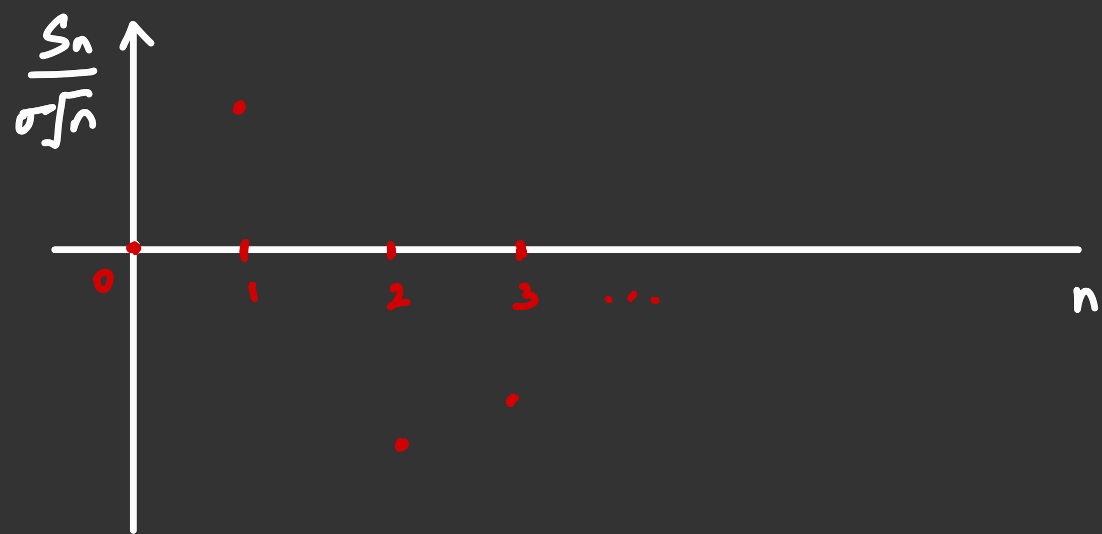
INDEPENDENCE.

EXISTENCE OF WIENER MEASURE.

LET $(\xi_1, \xi_2, \dots)$ BE AN IID SEQUENCE OF $\mathbb{R}$-VALUED RV'S.

ASSUME $\mathbb{E}|\xi_1|^2 = \sigma^2 < \infty$ & $E\xi_1 = 0$.

CONSIDER RANDOM WALK: $S_n := \sum_{i=1}^{n} \xi_i$, $S_0 = 0$.

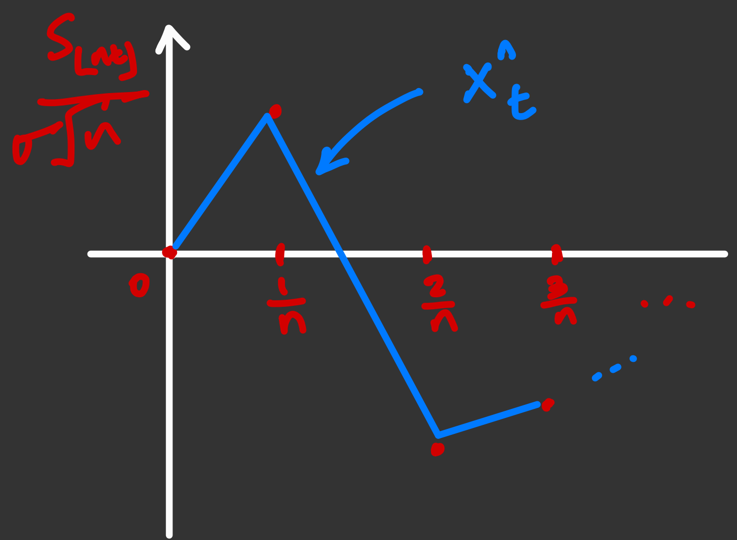


EXISTENCE OF WIENER MEASURE.

LET $(\xi_1, \xi_2, \ldots)$ BE AN IID SEQUENCE.

ASSUME $\mathbb{E}|\xi_1|^2 = \sigma^2 < +\infty$ & $E\xi_1 = 0$.

CONSIDER RANDOM WALK: $S_n := \sum_{i=1}^{n} \xi_i$, $S_0 = 0$.



$$X_t^n := \underline{\frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor}} + (nt - \lfloor nt \rfloor) \frac{1}{\sigma\sqrt{n}} \xi_{\lfloor nt \rfloor + 1}$$

$(X^n_t : t \in [0,1])$ IS A LINEAR INTERPOLATION OF S_n.

$\Rightarrow X^n \in C$.

STEPS FOR PROVING EXISTENCE:

1. SHOW F.D.D.'S OF X^n CONVERGE TO SOME LIMIT THAT HAS PRECISELY THE F.D.D.'S OF HYPOTHESIZED W.

2. SHOW THAT $\{X^n\}$ IS TIGHT. THEN, BY PROKHOROV, A SUBSEQUENCE $\{X^{n_i}\}$ CONVERGES TO A LIMIT.

   $\Rightarrow$ FDD'S OF $\{X^{n_i}\}$ CONVERGE.

   SINCE FDD'S OF $\{X^n\}$ CONVERGE, IT FOLLOWS THE SUBSEQUENTIAL LIMIT MUST BE W.

STEP 1: FIX $t \in [0,1]$.

LET $\Psi_{n,t} := \frac{1}{\sigma\sqrt{n}} \xi_{\lfloor nt \rfloor + 1} (\lfloor nt \rfloor - nt)$

BY CHEBYSHEV'S INEQUALITY:

$$\mathbb{P}\left(|\Psi_{n,t}| > a \right) \leq \frac{\mathrm{Var}(\Psi_{n,t})}{a^2}$$

$$= \frac{1}{a^2} \cdot \frac{1}{n} (\lfloor nt \rfloor - nt)^2$$

$\Rightarrow \quad \Psi_{n,t} \Rightarrow 0$ as $n \to \infty$,

SINCE $\frac{\lfloor nt \rfloor}{n} \to t$ as $n \to \infty$.

BY THE CENTRAL LIMIT THEOREM:

$$\frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor} \Rightarrow \sqrt{t}\, \mathcal{N}(0,1). \quad \text{as} \quad n \longrightarrow \infty.$$

FINALLY, BY SLUTSKY'S THEOREM,

$$X^n_t \Rightarrow \sqrt{t}\, \mathcal{N}(0,1). \quad \text{as} \quad n \longrightarrow \infty.$$

$$\downarrow$$

$$X^n_t = \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor} + \Psi_{n,t}.$$

NEXT, CONSIDER $s \leq t$ &

$$\left(X^n_s ,\ X^n_t - X^n_s \right) = \frac{1}{\sigma\sqrt{n}} \left(S_{\lfloor ns \rfloor} ,\ S_{\lfloor nt \rfloor} - S_{\lfloor ns \rfloor} \right)$$

$$+ \left(\psi_{n,s} ,\ \psi_{n,t} - \psi_{n,s} \right)$$

BY THE SAME CONSIDERATIONS,

$$\left(\psi_{n,s} ,\ \psi_{n,t} - \psi_{n,s} \right) \Rightarrow 0 \quad \text{as } n \to \infty.$$

SINCE $S_{\lfloor ns \rfloor} \perp \left(S_{\lfloor nt \rfloor} - S_{\lfloor ns \rfloor} \right)$,

$$\frac{1}{\sigma\sqrt{n}} \left(S_{\lfloor ns \rfloor} ,\ S_{\lfloor nt \rfloor} - S_{\lfloor ns \rfloor} \right) \Rightarrow \left(\mathcal{N}(0,s), \mathcal{N}(0,t-s) \right).$$

BY SLUTSKY'S THEOREM:

$$\left(X^n_s ,\ X^n_t - X^n_s \right) \Rightarrow \left(\mathcal{N}(0,s) ,\ \mathcal{N}(0,t-s) \right)$$

as $n \to \infty$.

BY THE CONTINUOUS MAPPING THEOREM IT FOLLOWS THAT:

$$\left(X^n_s ,\ X^n_t \right) \Rightarrow \left(\mathcal{N}(0,s) ,\ \mathcal{N}(0,s) + \mathcal{N}(0,t-s) \right).$$

as $n \to \infty$.

STEP 2: PROVE TIGHTNESS.

CONFIRM THE FOLLOWING LEMMA: (P. 88)

<u>LEMMA</u> Suppose $\{\xi_n\}$ is a stationary sequence

& $$\lim_{\lambda \to \infty} \overline{\lim_{n \to \infty}} \lambda^2 P\left(\max_{k \leq n} |S_k| \geq \lambda \sigma \sqrt{n} \right) = 0.$$

Then $\{X^n\}$ is tight.

TO DO SO, USE ETEMADIS INEQUALITY.

ETEMADIS INEQUALITY:

LET $(\xi_1, \xi_2, \dots)$ BE IID. DEFINE $S_n = \sum_{i=1}^{n} \xi_i$.

THEN:

$$\mathbb{P}\left(\max_{1 \le k \le n} |S_k| \geq 3\alpha \right)$$

$$\leq 3 \max_{1 \le k \le n} \mathbb{P}\left(|S_k| \geq \alpha \right)$$

TO VERIFY THE LEMMA IT SUFFICES TO

SHOW:

$$\lim_{\lambda \to \infty} \overline{\lim_{n \to \infty}} \lambda^2 \cdot \max_{k \le n} \mathbb{P}\left(|S_k| \geq \lambda \sigma \sqrt{n} \right) = 0.$$

FOR EXISTENCE OF WIENER MEASURE, CHOOSE A SPECIFIC $\xi_1 \equiv \mathcal{N}(0,1)$.

THEN, $\frac{S_k}{\sqrt{k}} \sim \mathcal{N}(0,1)$.

NOW, BY CHEBYSHEV'S INEQUALITY:

$$\mathbb{P}(|\xi_1| \geq \lambda) \leq \frac{E N^4}{\lambda^4} = \frac{3}{\lambda^4}.$$

$$\therefore \quad \mathbb{P}\left(|S_k| \geq \lambda \sigma \sqrt{n}\right) \leq \frac{3}{\lambda^4 \sigma^4} . \; \forall k \leq n.$$

$$\left(\leq \mathbb{P}\left(|S_k| \geq \lambda \sigma \sqrt{k}\right)\right)$$

$$\Rightarrow \lim_{\lambda \to \infty} \overline{\lim_{n \to \infty}} \; \lambda^2 . \max_{k \leq n} \mathbb{P}\left(|S_k| \geq \lambda \sigma \sqrt{n}\right) = 0.$$

FINALLY, PROVE THE LEMMA.

<u>PROOF:</u> VERIFY THE ARZELA-ASCOLI CRITERION OF THEOREM 7.3

<u>THEOREM 7.3</u> A sequence of measures $\{P_n\}$ on $(C, \mathcal{C})$ are tight iff

(i) $\forall \eta > 0$, $\exists$ an $a \in \mathbb{R}$ & $n_0 \in \mathbb{N}$ s.t.,

$$P_n\left(x : |x(0)| \geq a\right) \leq \eta, \quad \forall n \geq n_0.$$

(ii) $\forall \epsilon > 0, \eta > 0$, $\exists$ $0 < \delta < 1$ & $n_0 \in \mathbb{N}$ s.t.

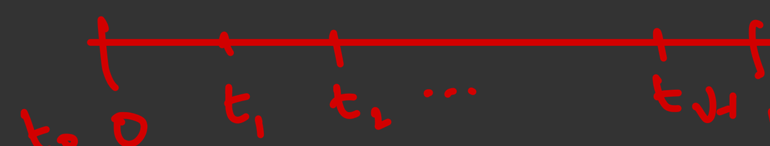
$$P_n\left(x : w_x(\delta) \geq \varepsilon\right) \leq \eta \quad \forall n \geq n_0.$$

TO SHOW CR. (ii), SUFFICES TO SHOW,

$$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} \mathbb{P}\left(w_{X^n}(\delta) \geq \epsilon \right) = 0 \quad \forall \varepsilon > 0.$$

BY THEOREM 7.4, FOR ANY $0 = t_0 < t_1 < \dots < t_v = 1$;

$$\mathbb{P}\left(w_{X^n}(\delta) \geq 3\epsilon \right)$$



$$\leq \sum_{i=1}^{v} \mathbb{P}\left(\sup_{t_{i-1} \leq s \leq t_i} \left| X^n_s - X^n_{t_{i-1}} \right| \geq \epsilon \right),$$

IF $t_i - t_{i-1} \geq \delta \quad \forall \ 1 < i < v.$

TAKE $\quad t_i = \dfrac{m_i}{n}$. FOR INTEGERS m_i & $m_v = n$.

THEN,

$$\mathbb{P}\left(w_{X^n}(\delta) \geq 3\epsilon \right)$$

$$\leq \sum_{i=1}^{v} \mathbb{P}\left(\max_{m_{i-1} \leq k \leq m_i} \frac{|S_k - S_{m_{i-1}}|}{\sigma\sqrt{n}} \geq \epsilon \right)$$

$$= \sum_{i=1}^{v} \mathbb{P}\left(\max_{k \leq m_i - m_{i-1}} |S_k| \geq \epsilon\sigma\sqrt{n} \right),$$

$\because$ S_k IS THE SUM OF STATIONARY RV'S.

(HOLDS PROVIDED $\dfrac{m_i}{n} - \dfrac{m_{i-1}}{n} \geq \delta$, $1 < i < v$.

CHOOSE $m_i = im$, $m = \lceil n\delta \rceil$ & $0 \le i < \vartheta$.

$\Rightarrow m = m_i - m_{i-1} \geq n\delta \cdot q$

$\vartheta = \left\lceil \frac{n}{m} \right\rceil \longrightarrow \frac{1}{\delta} < \frac{2}{\delta}$

$\frac{n}{m} \longrightarrow \frac{1}{\delta} > \frac{1}{2\delta}$

FOR LARGE n!

$$\mathbb{P}\left(w_{X^n}(\delta) \geq 3\epsilon \right) \leq \vartheta \cdot \mathbb{P}\left(\max_{k \leq m} |S_k| \geq \epsilon\sigma\sqrt{n} \right)$$

$$\leq \frac{2}{\delta} \mathbb{P}\left(\max_{k \leq m} |S_k| \geq \frac{\epsilon}{\sqrt{2\delta}} \sigma\sqrt{m} \right)$$

LET $\lambda = \frac{\epsilon}{\sqrt{2\delta}}$.

$$\Rightarrow \quad \mathbb{P}\left(w_{x^n}(\delta) \geq 3\epsilon \right) \leq \frac{4\lambda^2}{\epsilon^2} \mathbb{P}\left(\max_{k \leq m} |S_k| \geq \lambda \sigma \sqrt{m} \right)$$

Then by the Hypothesis it follows that $\exists\, \lambda$ s.t.

$$\lim_{\lambda \to \infty} \overline{\lim_{n \to \infty}} \lambda^2 \, \mathbb{P}\left(\max_{k \leq n} |S_k| \geq \lambda \sigma \sqrt{n} \right) = 0.$$

$$\frac{4\lambda^2}{\epsilon^2} \overline{\lim_{m \to \infty}} \, \mathbb{P}\left(\max_{k \leq m} |S_k| \geq \lambda \sigma \sqrt{m} \right) < \eta,$$

for given ϵ & η.

$\#$.

KOL. INEQ IS TOO LOOSE:

$$\mathbb{P}\left(\max_{k \le n} |S_k| \geq \lambda \sigma \sqrt{n} \right) \le \frac{27}{\lambda^2 \sigma^2 n} \cdot \mathrm{Var}(S_n)$$

$$= \frac{27}{\lambda^2 \cancel{\sigma^2} \cancel{n}} \cdot \cancel{n} \cancel{\sigma^2}$$

$$= \frac{27}{\lambda^2}.$$

$$\therefore \quad \lambda^2 \, \mathbb{P}(\cdots) \le 27.$$