Let's see the sense in which

Kolmogorov-Smirnov statistic

$$D_n := \sqrt{n}\,\| F_n - F \|_\infty$$

$$= \sqrt{n} \sup_{t \in \mathbb{R}} |F_n(t) - F(t)|$$

is "distribution free".

$$F_n(t) := \frac{1}{n} \sum_{j=1}^{t} \mathbb{I}(X_i \le t)$$

Suppose (i) $X_i \sim F$

(ii) F is continuous.

Then, for $t = F^{-1}(u)$, we can write

$$P\left(\overbrace{\sqrt{n} \sup_{t \in \mathbb{R}} \left| F_n(t) - F(t) \right|}^{D_n} \in A \right)$$

$$= P\left(\sqrt{n} \sup_{0 \le u \le 1} \left| F_n(F^{-1}(u)) - u \right| \in A \right) \qquad \text{——— (1)}$$

And, notice

$$F_n\left(F^{-1}(u)\right) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{I}\left(X_i \leq F^{-1}(u)\right)$$

$$= \frac{1}{n}\sum_{j=1}^{n} \mathbb{I}\Big(\underbrace{F(X_i)}_{U_i \text{ uniformly distributed}} \leq u\Big)$$

$$= G_n(u)$$

Therefore,

$$P\left(\sqrt{n}\sup_{t\in\mathbb{R}}\left|F_n(t) - F(t)\right| \in A\right)$$

$$= P\left(\sqrt{n}\sup_{0\leq u\leq 1}\left|G_n(u) - u\right| \in A\right).$$

$\boxtimes$

1. $\left\{ \sqrt{n}\,(G_n(u) - u),\ u \in [0,1] \right\}$ is called the the uniform empirical process.

2. The assumption that F is continuous is important.