Review: Weak Convergence

$$X_n \xrightarrow{d} X \quad \text{if}$$

(a) $P(X_n \in A) \longrightarrow P(X \in A)$

for P-continuity sets A

(b) If $f \in BC$, then

$$\mathbb{E}[f(X_n)] \longrightarrow \mathbb{E}[f(X)]$$

Review: $O_p(1)$

We say that a real-valued sequence $X_n = O_p(1)$ if for each $\varepsilon > 0$, $\exists$ $N(\varepsilon)$, $T(\varepsilon)$ s.t. $\forall$ $n \geq N(\varepsilon)$,

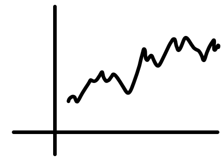
$$P\left(X_n \notin [-T(\varepsilon), T(\varepsilon)] \right) \leq \varepsilon.$$

We say $X_n = O_p(Y_n)$ if

$X_n / Y_n = O_p(1)$.

$\left(X_n = O_p(1) \equiv X_n \text{ is tight} \right)$

Review: Modulus of Continuity



$f : [0,1] \longrightarrow \mathbb{R}$.

The modulus of continuity

$$w(f, \delta) := \sup_{|x_1 - x_2| \le \delta} \left| f(x_1) - f(x_2) \right|,$$

$0 < \delta < 1$.

f is uniformly continuous if and only if $\lim_{\delta \to 0} w(f, \delta) = 0$.

Review: Totally Bounded

$(\mathcal{G}, \|\cdot\|)$ is totally bounded

if $H(\delta, \mathcal{G}, P) < \infty \quad \forall \delta > 0.$

(i) Recall Uniform LLN

$\mathcal{G}$ satisfies ULLN if

$$\sup_{g \in \mathcal{G}} \left| \int g \, dP_n - \int g \, dP \right| \xrightarrow{a.s.} 0$$

(If $G \in L_1(P)$ and $\frac{1}{n} H(\delta, \mathcal{G}, P_n) \xrightarrow{P} 0$ then $\mathcal{G}$ satisfies ULLN)

(ii) What is Uniform CLT?

<u>Classic CLT</u>: $\sqrt{n}(\bar{X} - \mu) \xrightarrow{d} N(0, \sigma^2)$

$$\nu_n := \left\{ \sqrt{n} \int g \, d(P_n - P), \, g \in \mathcal{G} \right\}$$

empirical process labeled by g.

For optimization

$$\nu_n := \left\{ \sqrt{n} \int F(x, Y)\, d(P_n - P), \quad x \in \mathcal{X} \right\}$$

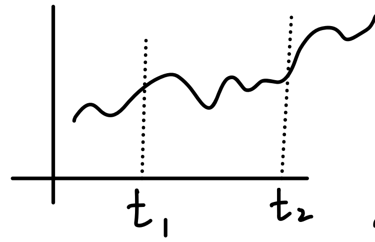
g (labeling is by x)

Does the process $\{\nu_n,\ n \geq 1\}$ converge to anything weakly?

?

$$\nu_n \xrightarrow{d} \nu$$

Gaussian Process



Check

Let $\nu := \{ \nu(g),\ g \in \mathcal{G} \}$

be a mean-zero process.

ν is a Gaussian process if the "marginal" is Gaussian and the covariance has a particular form:

(i) $g \sim$ Gaussian

(ii) $$\mathrm{Cov}\big(\nu(g_1), \nu(g_2)\big) = \int g_1 g_2 - \underbrace{\int g_1 \int g_2}_{0}$$

Uniform CLT or P-Donsker

The class $\mathcal{G}$ is <u>P-Donsker</u>

if $\nu_n \xrightarrow{d} \nu$ where

$$\nu_n := \left\{ \sqrt{n} \int g \, d(P_n - P), \; g \in \mathcal{G} \right\}$$

and ν is a <u>mean-zero</u>

<u>Gaussian process</u>.

What are some conditions?

Theorem (Dudley, 1984)

Suppose

(A) $(\mathcal{G}, \|\cdot\|)$ is totally bounded

(B) for each $\eta > 0$, $\exists\, \delta(\eta)$ s.t.

$$\lim_{n} P\left(\sup_{\|g_1 - g_2\| < \delta} \left| \nu_n(g_1) - \nu_n(g_2) \right| > \eta \right) < \eta .$$

Then, $\mathcal{G}$ is P-Donsker. ∎

Theorem

Suppose

(C) $G \in L_2(P)$

(D) $\exists\, H : [0,1] \to \mathbb{R}^+$ non-decreasing s.t.

(i) $\displaystyle\int_0^1 \sqrt{H(u)}\, du < \infty$

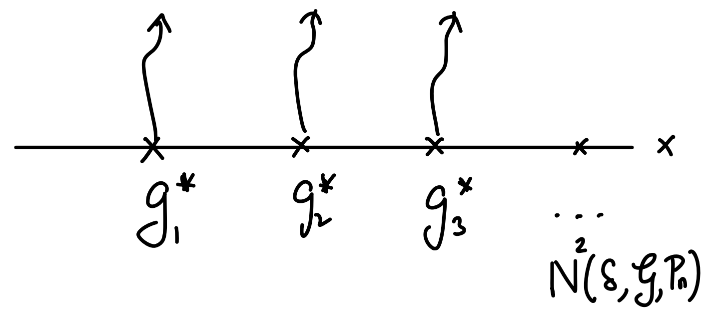
(ii) $\displaystyle\lim_{t\to\infty} \lim_{n} P\left(\sup_{\delta} \frac{H(\delta, \mathcal{G}, P_n)}{H(\delta)} > t \right) = 0$

emp entropy is uniformly $O_p(H)$.

Then, $\mathcal{G}$ is P-Donsker.

$\square$

Proof Sketch



Consider

$$\tilde{\mathcal{G}} := \left\{ g_1 - g_2 : g_1, g_2 \in \mathcal{G} \right\}$$

Notice

$$\tilde{g}_1 - \tilde{g}_2 = g_{11} - g_{12} - (g_{21} - g_{22})$$

$$H(\delta, \tilde{\mathcal{G}}, P_n) \leq 2 H(\delta/2, \mathcal{G}, P_n)$$

So, (C), (D) are satisfied for $\tilde{\mathcal{G}}$.

Now use Theorem 5.3 on the empirical process $\tilde{\nu}_n$ to see that (B) is satisfied.

Also, since $\tilde{G} \in L_2(P)$ and $\frac{1}{n} H(\delta, \tilde{\mathcal{G}}, P_n) \xrightarrow{p} 0$, $\tilde{\mathcal{G}}$ satisfies ULLN:

$$\sup_{g_1, g_2} \Big| \, \|g_1 - g_2\|_n - \|g_1 - g_2\| \, \Big| \xrightarrow{a.s.} 0$$

Therefore,

$$H(\delta, \mathcal{G}, P) \leq \overbrace{H(\delta/2, \mathcal{G}, P_n)}^{\text{finite wp1 because of (ii)}}$$

a.s. for large enough n.

Hence $\mathcal{G}$ is totally bounded. $\blacksquare$