σ-Field

Let Ω be an arbitrary set of points ω.

A class $\mathcal{F}$ of subsets of Ω is called a σ-field if

(i) $\Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$.

Probability Measure

A set function $P: \mathcal{F} \to [0, 1]$ is called a probability measure if

(i) $0 \le P(A) \le 1$ for $A \in \mathcal{F}$

(ii) $P(\Omega) = 1$

(iii) if $A_1, A_2, \ldots$ is a disjoint sequence of $\mathcal{F}$-sets,

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j)$$

Probability Space

If $\mathcal{F}$ is a σ-field in Ω and P is a probability measure on $\mathcal{F}$, then the triple $(\Omega, \mathcal{F}, P)$ is called a probability space.

Random Variable

A random variable (r.v.) is a real-valued measurable function on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that is, $X : \Omega \longrightarrow \mathbb{R}$ such that

$$X^{-1}((-\infty, x]) \in \mathcal{F} \ \forall x \in \mathbb{R}$$

Discrete Random Variable

X is called discrete r.v. if there exist countable distinct $x_1, x_2, \ldots \in \mathbb{R}$ s.t.

$$\sum_{j=1}^{\infty} P(X = x_j) = 1$$

CONTINUOUS RANDOM VARIABLE

If $P(X = x) = 0 \;\; \forall x \in \mathbb{R}$

then X is called a continuous random variable.

DENSITY FUNCTION

If there exists a function

$f : \mathbb{R} \to \mathbb{R}$ such that

the cumulative distribution

function (cdf)

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt, \quad \forall x \in \mathbb{R}$$

then f is called the probability density of X.

EXPECTATION

For measurable $g : \mathbb{R} \to \mathbb{R}$, and random variable X, $g(X)$ is a random variable.

If X is discrete with values $x_1, x_2, \ldots$

$$\mathbb{E}[g(X)] = \sum_{j=1}^{\infty} g(x_j)\, P(X = x_j)$$

provided the series converges absolutely.

If X has a density $p_X(\cdot)$,

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, p_X(x)\, dx$$

EXPECTATION

For a "general" r.v. X having cdf F_X and measurable $g: \mathbb{R} \to \mathbb{R}$, we write

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, dF_X(x)$$

Lebesgue-Stieltjes

Joint Distributions

Given a pair of r.v.s (X, Y),

the joint cdf

$$F_{X,Y}(x,y) = P(X \le x, Y \le y).$$

The marginals

$$F_X(x) = P(X \le x) = F_{X,Y}(x, \infty)$$

$$F_Y(y) = P(Y \le y) = F_{X,Y}(\infty, y)$$

Joint Distributions

(X, Y) has a density $p_{X,Y}$

means

$$F_{X,Y}(x, y) = \int_{-\infty}^{y}\int_{-\infty}^{x} p_{X,Y}(s, t)\, ds\, dt$$

If X and Y are independent

$p_{X,Y}(\cdot, \cdot)$ is necessarily of the

form $p_X(\cdot)\, p_Y(\cdot)$.

$\boxed{\text{CONDITIONAL DISTRIBUTIONS}}$

Given events A, B

$$P(A \mid B) := \frac{P(A \cap B)}{P(B)}$$

$$F_{X|Y}(x \mid y) = P(X \leq x \mid Y = y)$$

$$= \frac{P(X \leq x, Y = y)}{P(Y = y)}.$$

if $P(Y=y) > 0$.

$\boxed{\text{CONDITIONAL DISTRIBUTIONS}}$

$$F_{X|Y}(x \mid y) = \frac{P(X \le x, Y = y)}{P(Y = y)}$$

$$= \frac{\int_{-\infty}^{x} p_{X,Y}(s, y)\, ds}{\int_{-\infty}^{\infty} p_{X,Y}(s, y)\, ds}$$

$$P(X \le x, Y \le y) = \int_{-\infty}^{y} F_{X|Y}(x \mid t)\, dF_Y(t)$$

CONDITIONAL DISTRIBUTIONS

NOTICE :

(i) $F_{X|Y}(x|y)$ is a cdf in x for each y.

(ii) $F_{X|Y}(x|y)$ is a function of y for each x

$\boxed{\text{LAW OF TOTAL PROB.}}$

$$P(X \le x) = P(X \le x, Y \le \infty)$$

$$= \int_{-\infty}^{\infty} F_{X|Y}(x \mid t)\, dF_Y(t)$$

CONDITIONAL DENSITY

If X, Y are jointly distributed conditional r.v.s, the conditional density

$$p_{X|Y}(x|y) = \frac{d}{dx} F_{X|Y}(x|y)$$

$$= \frac{p_{X,Y}(x,y)}{p_Y(y)},$$

$p_Y(y) > 0.$

CONDITIONAL EXPECTATION

Suppose $g : \mathbb{R} \to \mathbb{R}$ is a function for which $\mathbb{E}[g(X)] < \infty$.

Then,

$$\mathbb{E}[g(X) \mid Y = y] = \int_{-\infty}^{\infty} g(x) \, dF_{X|Y}(x|y)$$

CONDITIONAL EXPECTATION

For any bounded function

$h : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}\left[g(X)\,h(Y)\right]$$

$$= \int_{-\infty}^{\infty} \mathbb{E}\left[g(X) \,\middle|\, Y = y\right] h(y)\, dF_Y(y)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)\, dF_{X|Y}(x|y)\, h(y)\, dF_Y(y)$$

FINITE DIMENSIONAL DISTBNS

Let $\{X_n, n \geq 1\}$ be a sequence of r.v.s. The joint distributions $F_{X_{i_1}, X_{i_2}, \ldots X_{i_n}}(x_{i_1}, x_{i_2}, \ldots x_{i_n})$, $i_1, i_2, \ldots, i_n \in \mathbb{N}$ and distinct

are called the finite-dimensional distributions associated with $\{X_n, n \geq 1\}$.

LIMIT

Suppose $\{\alpha_n,\ n \geq 1\}$ is a sequence of reals.

$$\lim_{n\to\infty} \alpha_n = \alpha$$

means for given $\varepsilon > 0$, $\exists\, N(\varepsilon)$ s.t. for all $n \geq N(\varepsilon)$

$$|\alpha_n - \alpha| \leq \varepsilon.$$

LIMIT OF A RANDOM SEQ

Let $\{Z_n, n \geq 1\}$ be a sequence of random variables.

(a) Convergence with prob. one.

$$Z_n \longrightarrow Z \quad a.s.$$

means

$$P\left(\omega : \lim_n Z_n(\omega) = Z(\omega)\right) = 1.$$

LIMIT OF A RANDOM SEQ

(b) Convergence in probability

$$Z_n \xrightarrow{p} Z$$

means, for every $\varepsilon > 0$,

$$\lim_n P\left(\omega : \left| Z_n(\omega) - Z(\omega) \right| > \varepsilon\right) = 0$$

LIMIT OF A RANDOM SEQ

(c) Convergence in L_2

$$Z_n \xrightarrow{L_2} Z$$

means,

$$\lim_n \mathbb{E}\left[|Z_n - Z|^2 \right] = 0.$$

LIMIT OF A RANDOM SEQ

(d) Convergence in Distbn.

$$Z_n \xrightarrow{d} Z$$

means

$$F_n(t) \longrightarrow F(t)$$

for all F-continuity pts t,

where Z_n has distbn F_n and Z has distbn F.

LIMIT OF A RANDOM SEQ

$$
\begin{tikzcd}
\text{a.s. conv} \arrow[rr, Leftrightarrow, "\times" description] \arrow[dr, Rightarrow] & & L_2 \text{ conv} \arrow[dl, Rightarrow] \\
& \text{in prob. conv} \arrow[d, Rightarrow] & \\
& \text{in distbn conv} &
\end{tikzcd}
$$

STRONG LAW

Let $\{X_n, n \geq 1\}$ form an iid sequence of r.v.s. with $\mathbb{E}[X_1] = \mu \in \mathbb{R}$

Then

$$\bar{X}_n \longrightarrow \mu \quad a.s.,$$

where $\bar{X}_n := \frac{1}{n}(X_1 + X_2 + \cdots X_n).$

WEAK LAW

Let $\{X_n, n \geq 1\}$ form an iid sequence of r.v.s. with $\mathbb{E}[X_1] = \mu \in \mathbb{R}$.

Then

$$\bar{X}_n \xrightarrow{p} \mu$$

where $\bar{X}_n := \frac{1}{n}(X_1 + X_2 + \cdots X_n)$.

CENTRAL LIMIT THM.

Let $\{X_n, n \geq 1\}$ form an iid sequence of r.v.s. with $\mathbb{E}[X_1] = \mu \in \mathbb{R}$ and $\mathrm{Var}(X_1) = \sigma^2 < \infty$. Then,

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} Z$$

where $Z \stackrel{d}{=} N(0,1)$, $S_n := \sum_{j=1}^{n} X_j$.

BOREL CANTELLI LEMMAS

FIRST BC LEMMA:

If $\{A_n, \; n \geq 1\}$ are events

such that

$$\sum_{n=1}^{\infty} P(A_n) < \infty$$

lim sup A_n

then the event

$$\{A_n \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

happens with probability zero.

BOREL CANTELLI LEMMAS

SECOND BC LEMMA:

If $\{A_n, n \geq 1\}$ are <u>independent</u> events such that

$$\sum_{n=1}^{\infty} P(A_n) = \infty$$

then the event

$$\{A_n \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

happens with probability one.

MARKOV'S INEQ.

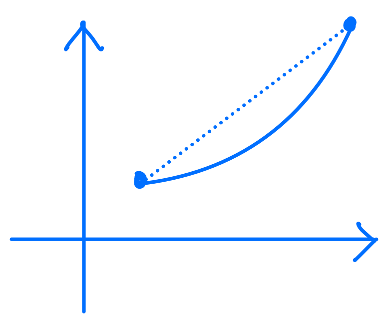
Let Z be a non-negative random variable. Then,

$$P(Z > t) \leq \frac{\mathbb{E}[Z]}{t}, \quad t>0.$$

JENSEN'S INEQ.

Let Z be a random variable and let $\psi : \mathbb{R} \to \mathbb{R}$ be a convex function. Then,

$$\mathbb{E}\left[\psi(Z)\right] \geq \psi\left(\mathbb{E}[Z]\right)$$



CAUCHY SCHWARZ

Suppose X and Y are r.v.s with $\mathbb{E}[X^2] < \infty$, $\mathbb{E}[Y^2] < \infty$. Then,

$$\mathbb{E}[XY] \leq \left(\mathbb{E}[X^2]\right)^{1/2} \left(\mathbb{E}[Y^2]\right)^{1/2}$$

Hölder's Inequality:

$$\mathbb{E}[|XY|] \leq \left(\mathbb{E}[|X|^p]\right)^{1/p} \left(\mathbb{E}[|Y|^q]\right)^{1/q}$$

for $p, q \in [1, \infty]$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$.