Stochastic Processes from the natural paradigm to study random phenomena that evolve over space and time.

The obvious intent and implication is the ability to describe and predict the functioning of a physical process of interest.

The course treats the mathematical machinery needed for such study.

A stochastic process $\{X_t, t \in T\}$ is a collection of random variables, taking values in $S \subseteq \mathbb{R}^d$, that is, $X_t \in S$.

T is called the <u>index set</u>, e.g.,

$T = (-\infty, \infty)$, $T = \{0, 1, 2, \ldots\}$

S is called the <u>state space</u>

A crucial element of stochastic processes is specifying, explicitly or implicitly, the relationship between X_t, $t \in T$.

When is a stochastic process $\{X_t, t \in T\}$ well specified?

Recall: A random variable X is a real measurable function on the probability space $(\Omega, \mathcal{F}, P)$, that is,

$$X^{-1}A := \{\omega \in \Omega : X(\omega) \in A\} \in \mathcal{F}.$$

To a large extent, we worked with the cumulative distribution function, which for a random variable X is

$$F(x) := P\left(X^{-1}(-\infty, x]\right).$$

A random variable is not uniquely specified by its cdf, although given a cdf one can construct a random variable on an appropriate probability space.

A stochastic process $\{X_t, t \in T\}$ is <u>well-defined</u> <u>or specified</u> (for purposes of this course) when we specify the following.

(i) the state space S

(ii) the time index T.

(iii) all <u>finite-dimensional</u> <u>joint distributions</u>, that is, the joint distributions of all vectors of the type

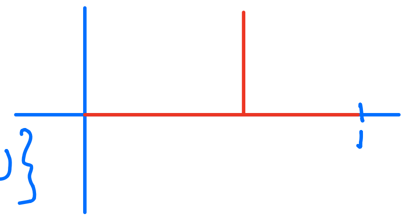
$$(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$$

where $t_1, t_2, \ldots, t_n \in T$, $n \in \mathbb{N}$.

(i), (ii), (iii) may not always suffice.

The issue is with (iii).

<u>Example</u>



$$X_t := \begin{cases} 0 & t \in [0,1] \setminus \{U\} \\ 1 & U = t \end{cases}$$

where $U \sim \text{Uniform}(0,1)$ r.v.

$$Y_t := 0 \quad \forall \; t \in [0,1]$$

We can check that X_t and Y_t have the same finite dimensional distributions.

However, $P(X_t \leq 1/2 \;\; \forall \; t) = 0$

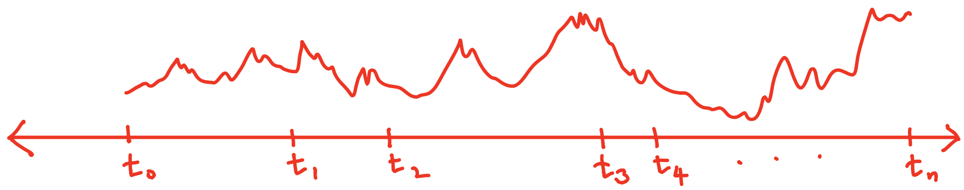
$P(Y_t \leq 1/2 \;\; \forall \; t) = 1$

Particular stochastic processes are specified for a context based on whether specific properties are satisfied. Let's look at some.

(i) Independent Increments
(ii) Stationary Increments
(iii) Martingale
(iv) Markov
(v) Stationarity

P1. Independent Increments

(e.g., Brownian motion, Poisson process, random walks)



$\{X_t, t \in T\}$ is said to have **independent increments** if the increments exhibited on disjoint intervals are independent, that is,

for $t_0 < t_1 < t_2 < \cdots t_n \in T$,

$$X_{t_1} - X_{t_0}, \; X_{t_2} - X_{t_1}, \; \ldots, \; X_{t_n} - X_{t_{n-1}}$$

are independent.

If t_0 is the smallest element in T, then

$$X_{t_0}, \; X_{t_1} - X_{t_0}, \; \ldots, \; X_{t_n} - X_{t_{n-1}}$$

are independent.

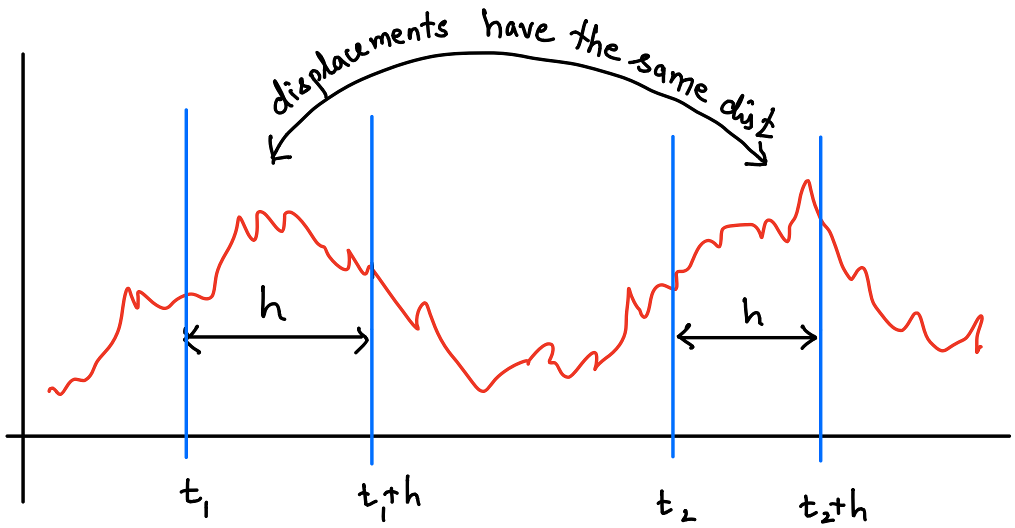
P2. Stationary Increments (e.g., homogeneous Poisson process)

$\{X_t, t \in T\}$ is said to have stationary increments if the distribution of

$X_{t_1+h} - X_{t_1}$ and $X_{t_2+h} - X_{t_2}$

depends only on h (for any $t_1, t_2 \in T$), that is,

$$P(X_{t_1+h} - X_{t_1} \in A) = P(X_{t_2+h} - X_{t_2} \in A)$$

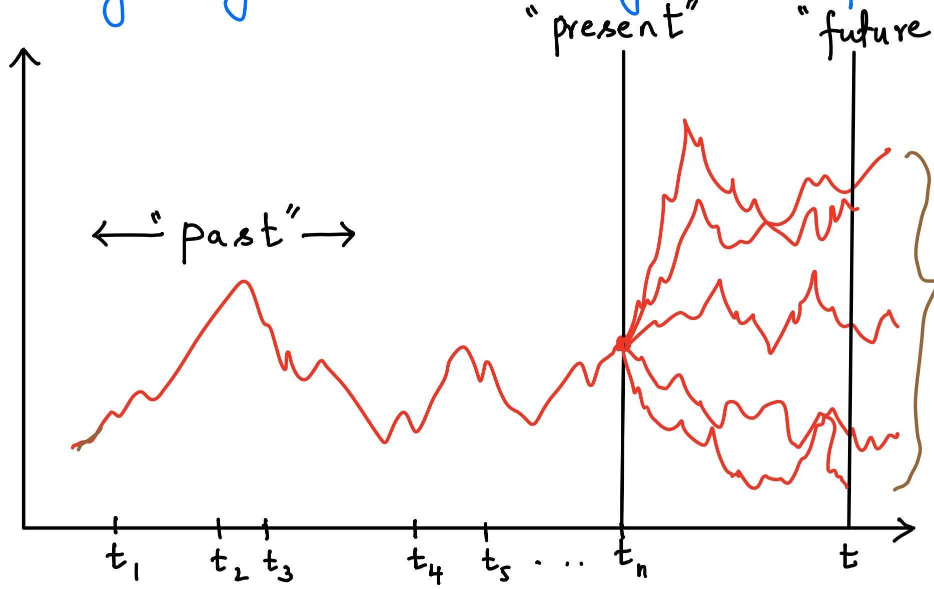


P3. Martingale (e.g., Brownian motion with zero drift, random walk with zero drift)

$\{X_t, t \in T\}$ is said to be a martingale if X_t is a "fair game" that is, $\mathbb{E}[|X_t|] < \infty \ \forall t$ and for any $t_1 < t_2 < \cdots < t_n \leq t \in T$

$$\mathbb{E}\left[X_t \ \middle|\ X_s = a_s \ s \in \{t_1, t_2, \ldots t_n\} \right] = a_{t_n} .$$

On average, you add nothing in the future!



P4. Markov Process

$\{X_t, t \in T\}$ is said to be Markov if the "future depends only on the present, but not how we got to the present."

$$P(X_t \in A \mid X_{t_1} = x_1, X_{t_2} = x_2, \ldots, X_{t_n} = x_n)$$
$$= P(X_t \in A \mid X_{t_n} = x_n)$$

whenever $t > t_n > t_{n-1} \cdots > t_1$ and appropriate sets A.

A Markov process with a countable state space is called a <u>Markov chain</u>.

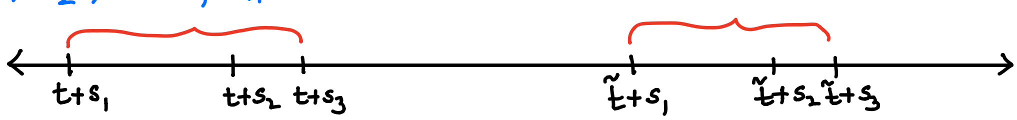
A Markov process with continous sample paths $\{X_t, t \in [0, \infty)\}$ is called a <u>diffusion</u>.

P5. Stationary Processes

A stochastic process is said to be **strictly stationary** if the joint distribution of

$$\left(X_{t+s_1}, X_{t+s_2}, \ldots, X_{t+s_n} \right)$$

is the same for all t and arbitrary selection of $s_1, s_2, \ldots, s_n$.



A stochastic process is said to be **covariance stationary** if it has **finite second moments** and

$$\mathrm{Cov}\left(X_{t+h}, X_t \right)$$

depends only on h for all t.

1. Are Markov processes stationary?
2. Are strictly stationary processes cov. stationary?
3. Give an example of a martingale that is not Markov

What sufficient condition ensures that $\{X_t, t \in T\}$ is well-specified through finite-dimensional distributions?

$$\lim_{\tau \to t} P\left(|X_t - X_\tau| > \varepsilon\right) = 0, \ \forall \varepsilon, t$$

(Notice that if the path X_t is continuous in t, the above is satisfied due to BCT.)