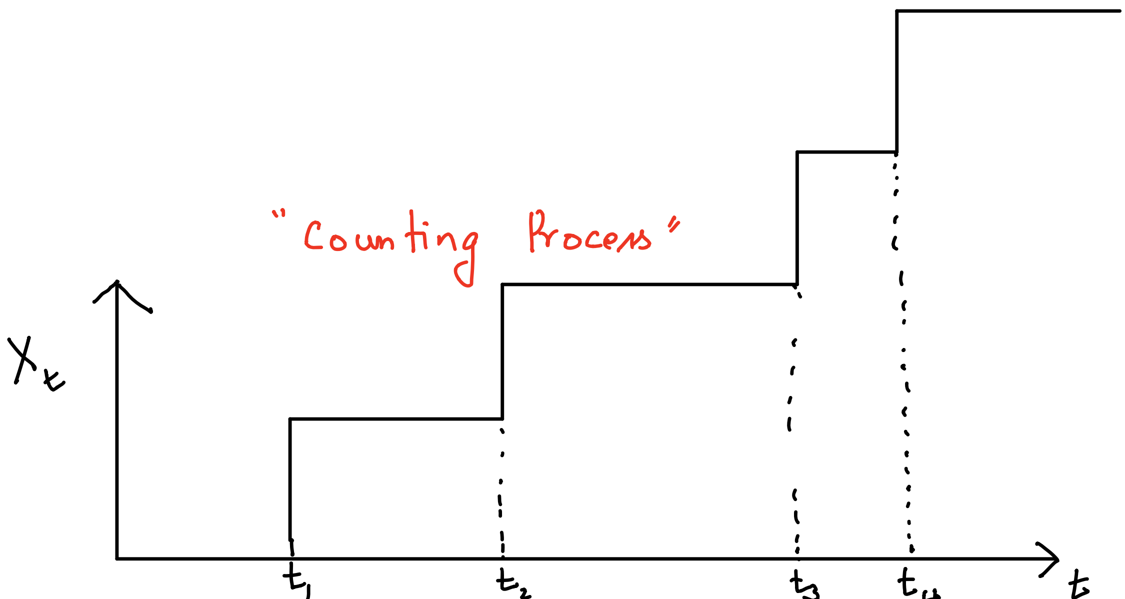
Example I. Poisson Process.

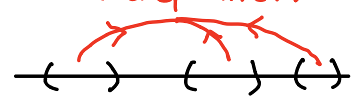
$\{X_t,\ t \geq 0\}$, $X_t \in \mathbb{N}$. $X_0 = 0$

"Counting Process"

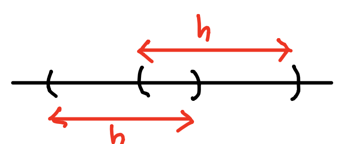


Specified by three postulates:

(a) Independent Increments



(b) Stationary Increments



incr. dist. depends only on length

(c) $P(X_{t+h} - X_t \geq 1) = \lambda h + o(h)$, $\lambda > 0$

$P(X_{t+h} - X_t \geq 2) = o(h)$.

X_t : No of "events" in $[0, t]$,

$t \geq 0$.

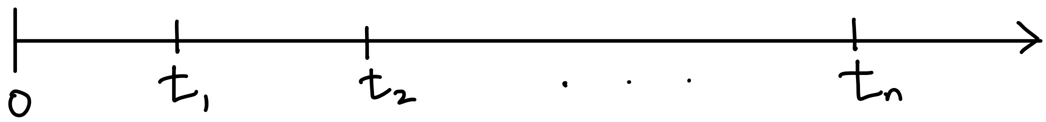
$\boxed{\text{Thm. 1.1}}$ [Distribution of X_t]

$$P(X_t = m) = \frac{e^{-\lambda t} (\lambda t)^m}{m!},$$

$m = 0, 1, 2, \ldots$

$\blacksquare$

X_t is distributed as Poisson(λt).



$\boxed{\text{Thm. 1.2}}$ [Distribution of $(X_{t_1}, X_{t_2} \dots X_{t_n})$]

Suppose $0 = t_0 < t_1 < t_2 \dots < t_n < \infty$.

Then,

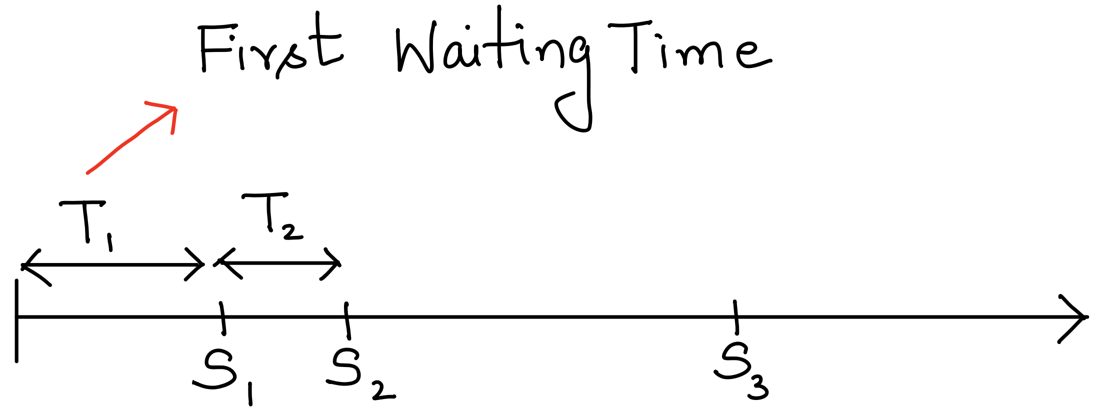
$$P(X_{t_1} = m_1, X_{t_2} = m_2, \dots X_{t_n} = m_n)$$

$$= \prod_{j=1}^{n} \frac{e^{-\lambda \Delta_j} (\lambda \Delta_j)^{m_j - m_{j-1}}}{(m_j - m_{j-1})!},$$

$$\Delta_j := t_j - t_{j-1}, \quad j = 1, 2, \dots, n$$

and $0 \le m_1 \le m_2 \le \dots \le m_n < \infty$

$\blacksquare$



Thm. 1.3 [Waiting Time Distbn.]

T_k, $k \geq 1$ are iid with

$$P(T_k > x) = e^{-\lambda x}, \quad x \geq 0$$

∎

The process restarts itself continuously due to the memoryless prop. of the exponential.

A useful result when proving Thm. 3.

$\boxed{\text{Lemma } 1\cdot 4}$

If $G : \mathbb{R}^+ \to [0, 1]$ is a non-increasing function such that $G(0) = 1$ and $\exists$ x s.t. $G(x) > 0$, then $G(x) = e^{-\lambda x}$ (for some $\lambda > 0$) if and only if

$$G(x+y) = G(x)\,G(y), \quad x \geq 0, y \geq 0.$$

$\boxed{\text{Proof Sketch of Thm 1.3}}$

We will adopt the following path.

1. Identify the density fn of $(S_1, S_2, \dots, S_k)$

2. Since

$$T_1 = S_1$$

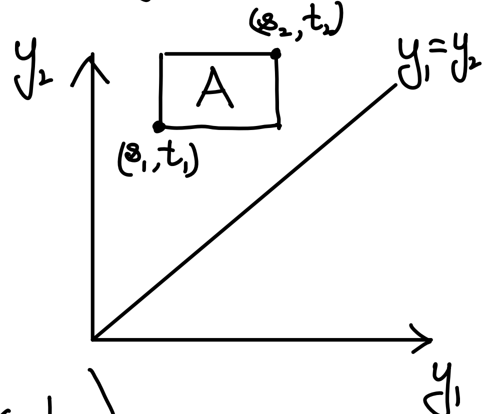
$$T_2 = S_2 - S_1$$

$$T_3 = S_3 - S_2$$

$$\vdots$$

We can use the density of $(S_1, S_2, \ldots, S_k)$ to derive the joint density of $(T_1, T_2, \ldots, T_k)$.

Lets perform Step 1 for (S_1, S_2).



$$P(s_1 \le S_1 \le t_1,\ s_2 \le S_2 \le t_2)$$

$$= P(X_{s_1} = 0,\ X_{t_1} - X_{s_1} = 1,\ X_{s_2} - X_{t_1} = 0,\ X_{t_2} - X_{s_2} \ge 1)$$

$$= e^{-\lambda s_1}\, \lambda(t_1 - s_1) e^{-\lambda(t_1 - s_1)}\, e^{-\lambda(s_2 - t_1)}$$

$$\times \left(1 - e^{-\lambda(t_2 - s_2)}\right)$$

$$= \lambda (t_1 - s_1)\left(e^{-\lambda s_2} - e^{-\lambda t_2}\right)$$

$$= \int_{s_2}^{t_2} \int_{s_1}^{t_1} \lambda^2 e^{-\lambda y_2} \, dy_1 \, dy_2$$

So, for any rectangle A in $\{(y_1, y_2) : 0 \leq y_1 < y_2\}$

$$P\left((S_1, S_2) \in A\right) = \iint_A \lambda^2 e^{-\lambda y_2} dy_1 dy_2$$

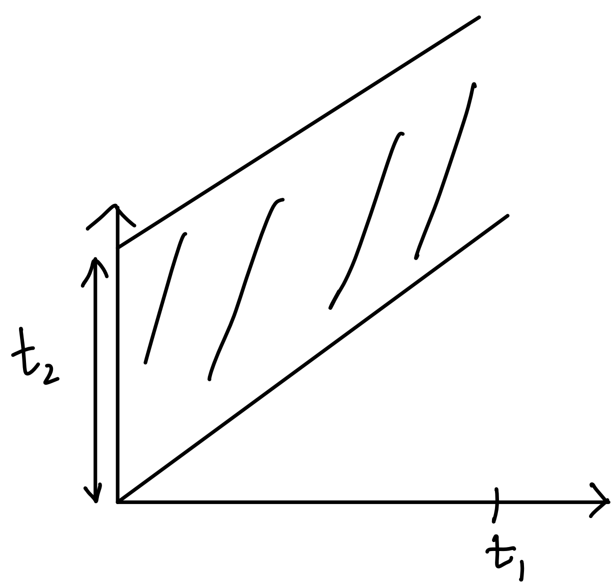
So, $f_{(S_1,S_2)}(y_1, y_2) = \lambda^2 e^{-\lambda y_2}$, $0 < y_1 < y_2 < \infty$. —— (1)

Let's now perform step 2.

$$(T_1, T_2) \equiv (S_1, S_2 - S_1)$$

$$P(T_1 \le t_1, T_2 \le t_2)$$

$$= \int_0^{t_1} \int_{y_1}^{t_2 + y_1} f_{(S_1, S_2)}(y_1, y_2)\, dy_2\, dy_1$$



$\boxed{\text{Beware!}}$

The proof of Thm. 1.3 in many textbooks at least incomplete, if not wrong.

$\boxed{\text{Thm. } 1\cdot 5}$ [CONDITIONAL DISTBN]

For any numbers s_j, $j = 1, 2, \ldots, n$ satisfying $0 \le s_1 \le s_2 \cdots s_n \le t$

$$P\left(S_1 \le s_1, S_2 \le s_2, \ldots, S_n \le s_n \mid N(t) = n\right)$$

$$= \frac{n!}{t^n} \int_0^{s_1} \cdots \int_{x_{n-2}}^{s_{n-1}} \int_{x_{n-1}}^{s_n} dx_n \, dx_{n-1} \cdots dx_1 ,$$

which is the distribution of the order statistics from a sample of n observations taken from $U(0, t)$.

Thm. 1.5 says that the conditional distribution of arrival times is the same as "uniform order statistics" in $[0, t]$.

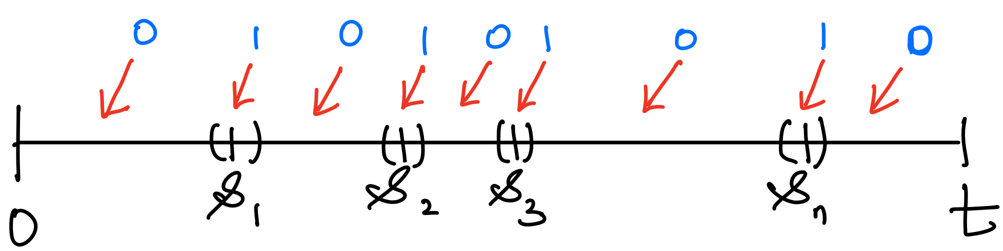
Can you prove?

PROOF SKETCH OF THM. 1.5

$$\text{L.H.S} = \frac{P\left(\bigcap_{j=1}^{n} S_j \leq s_j,\ N(t) = n\right)}{P(N(t) = n)}$$

"Heuristic" Proof

Let's find the density associated with the numerator.



$$dF_{S_j,\,1\le j\le n;\,N(t)}(s_1, s_2, \ldots, s_n, n) = \left(\prod_{j=1}^{n} e^{-\lambda s_j} \lambda\, ds_j\right) e^{-\lambda(t-s_n)}$$

$$= \lambda^n e^{-\lambda t} ds_1\, ds_2 \cdots ds_n$$

and hence,

$$f_{S_1, S_2, \ldots, S_n}(s_1, s_2, \ldots, s_n)$$

$$= \frac{\lambda^n e^{-\lambda t}}{P(N(t) = n)}$$

$$= \frac{n!}{t^n}, \quad 0 < s_1 < s_2 < \cdots s_n < t$$

For a rigorous proof of Thm 1.5, use the proof technique of Thm. 1.3.

∎

Recall the postulates of the (homogeneous) Poisson process.

(P.1) $X_0 = 0$;

(P.2) independent increments;

(P.3) (a) $P\left(X_{t+h} - X_t \geq 1 \mid X_t = x\right)$

$$= \lambda h + o_1(h)$$

(b) $P\left(X_{t+h} - X_t \geq 2 \mid X_t = x\right)$

$$= o_2(h).$$

By modifying postulate (P.3), we can arrive at very useful generalizations of the Poisson process.

Let's look at two

(i) Pure birth process

(ii) NHPP (Non Hom. Poisson Proc.)

PURE BIRTH PROCESS

(P.1) $X_0 = 0$;

(P.2) independent increments;

(P.3) (a) $P(X_{t+h} - X_t \geq 1 \mid X_t = k)$

$$= \lambda_k h + o_{1,k}(h)$$

(b) $P(X_{t+h} - X_t \geq 2 \mid X_t = k)$

$$= o_{2,k}(h) .$$

NOTICE

- Notice the "state dependent" rate
- Does the process retain stationary increments?
- Very useful to model population growth, etc.

Through similar calculations,

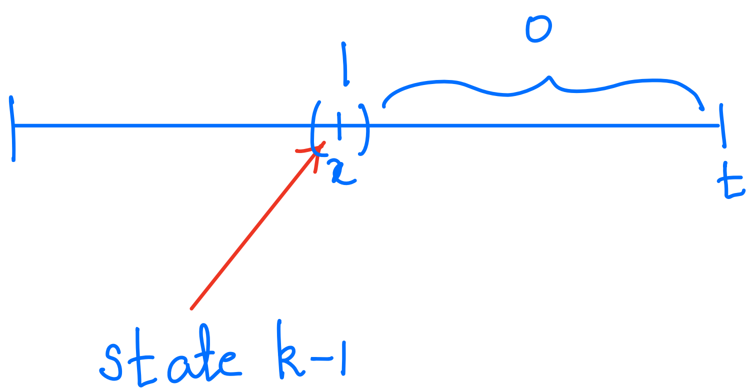
$$P_0'(t) = -\lambda_0 P_0(t)$$

$$P_m'(t) = -\lambda_m P_m(t) + \lambda_{m-1} P_{m-1}(t)$$

$$P_0(0) = 1 \qquad P_m(0) = 0, \ m \geqslant 1$$

Solve to get :

$$P_k(t) = \int_0^t e^{-\lambda_k (t-x)} \lambda_{k-1} P_{k-1}(x)\, dx$$



NON-STATIONARY POISSON PROCESS

(NSP.1) $X_0 = 0$;

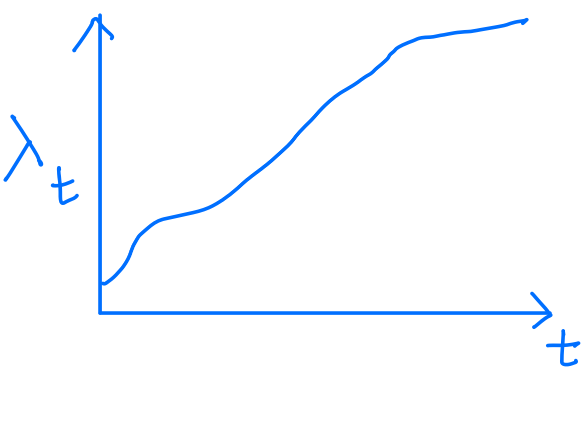
(NSP.2) independent increments;

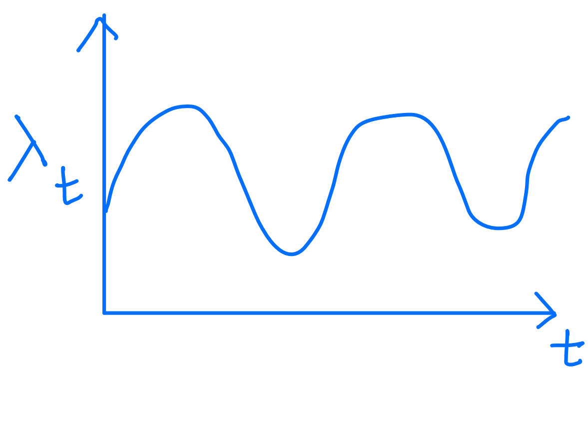
(NSP.3) (a) $P\left(X_{t+h} - X_t \geq 1\right)$

$$= \lambda(t)\,h + o_1(h)$$

(b) $P\left(X_{t+h} - X_t \geq 2\right)$

$$= o_2(h).$$





Thm. 1.6 Suppose the counting process $\{X_t, t \geq 0\}$ satisfies (NsP.1) — (NsP.3). Suppose also that $\lambda(\cdot)$ is continuous in $[0, t]$. Then,

$$P(X_t = k) = \frac{e^{-\int_0^t \lambda(x)\,dx} \left(\int_0^t \lambda(x) \right)^k}{k!}$$

$$k = 0, 1, 2, \ldots$$

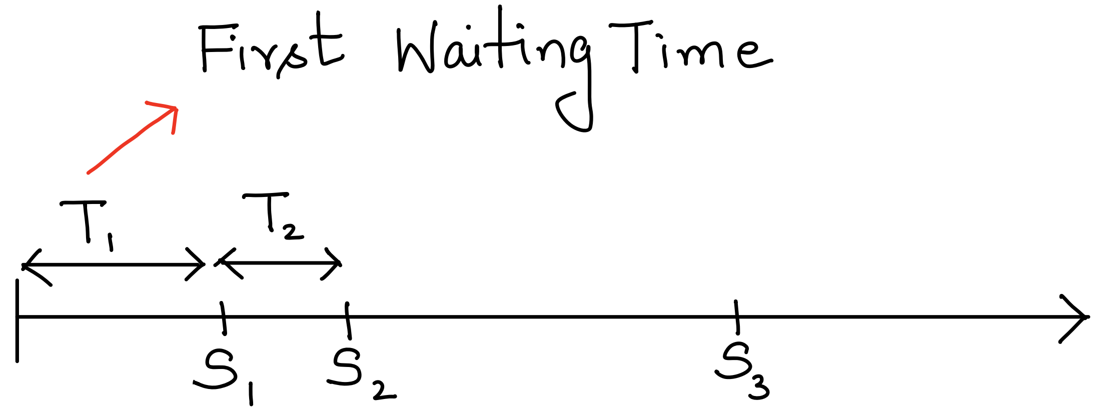
In fact, for any interval $(s, s']$

$$P(X_{s'} - X_s = k)$$

$$= \frac{e^{-\int_s^{s'} \lambda(x)\,dx} \left(\int_s^{s'} \lambda(x)\,dx \right)^k}{k!}.$$

$k = 0, 1, 2, \ldots$

How do the inter-arrival times of a non-stationary Poisson process behave?



- they cannot be independent
- are they exponential?

$\boxed{\text{Thm. 1.7}}$ Suppose $\{X_t, t \geq 0\}$

satisfies (NsP.1) — (NsP.3).

with $\Lambda(t) = \mathbb{E}[X_t]$ continuous in $t \geq 0$.

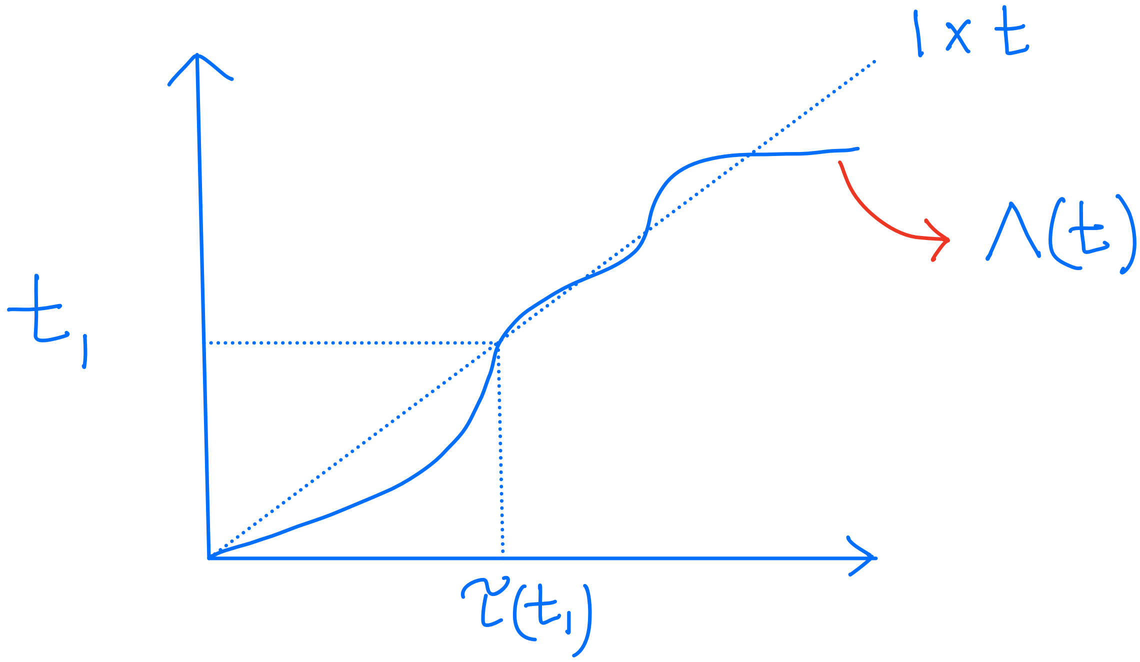
Then,

$$P(S_{n+1} - S_n > t \mid S_1, S_2, \ldots, S_n)$$

$$= \exp\left\{ -\int_{S_n}^{S_n+t} \lambda(x)\,dx \right\}$$

$\blacksquare$

Notice that Thm 1.7 says the waiting times are exponential but neither independent nor identically distributed.



$$\Lambda(t) := \int_0^t \lambda(x)\, dx$$

$\mathbb{E}[X_t]$

$$\tau(t) := \Lambda^{-1}(t)$$

$$:= \min\{s : \Lambda(s) \geq t\}$$

$t \geq 0.$

$\boxed{\text{Thm. 1.8}}$ Suppose $\{X_t, t \geq 0\}$ satisfies (NsP.1) — (NsP.3). with $\Lambda(t) = \mathbb{E}[X_t]$ continuous in $t \geq 0$.

Define:

$$Y_t = X_{\tau(t)}, \quad t \geq 0.$$

Then $\{Y_t, t \geq 0\}$ is a stationary Poisson process with rate $\lambda = 1$.

$\blacksquare$

$\boxed{\text{Proof Sketch}}$

Lets check whether $\{Y_t, t \geq 0\}$ satisfies the postulates

NsP.1 — NsP.3:

1. $Y_0 = 0$ ✓ since $\tau(0)=0$
2. independent increments ✓

   since $t_2 > t_1 \Rightarrow \tau(t_2) > \tau(t_1)$.
3. Notice:

$$P(Y_{t+h} - Y_t \geq 1)$$

$$= 1 - P(Y_{t+h} - Y_t = 0)$$

$$= 1 - P\left(X_{\tau(t+h)} - X_{\tau(t)} = 0\right)$$

$$= 1 - e^{-\left(\Lambda(\tau(t+h)) - \Lambda(\tau(t))\right)}$$

$$= 1 - e^{-\left(\Lambda(\Lambda^{-1}(t+h)) - \Lambda(\Lambda^{-1}(\tau(t)))\right)}$$

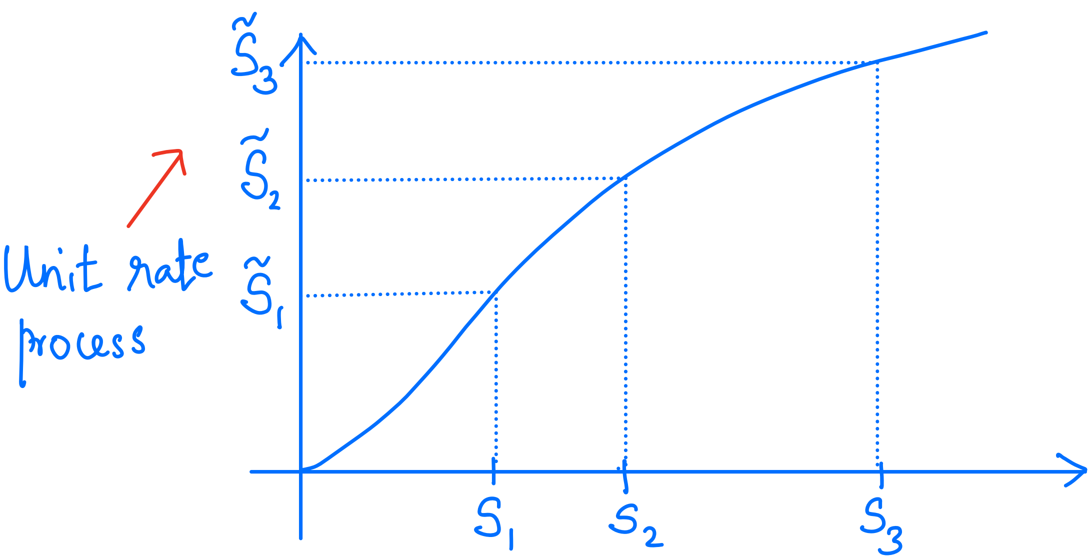
$$= 1 - e^{-h} = h + o(h).$$

$\square$

$\boxed{\text{Thm. 1.9}}$

$S_1, S_2, \dots,$ are arrival times of a non-stationary Poisson process with cont. cumulative rate $\Lambda(t), t \geq 0$ <u>if and only if</u> $\Lambda(S_1), \Lambda(S_2), \dots$ are arrival times of a stationary Poisson process with rate $\lambda = 1$. ■

A proof of Thm. 1.9 follows in a trivial way from Theorem 1.8.



Thm. 1.9 gives an effective method to simulate arrivals from any nonstationary Poisson process having (integrable) rate $\lambda(\cdot)$.

$n = 0, \; S_0 = 0$

while $S_n \leq t$ do

$\quad n = n+1$

$\quad$ independently generate $U_n \sim \text{Unif}(0,1)$

$$\tilde{T}_n = -\ln U_1$$

$$\tilde{S}_n = \tilde{S}_{n-1} + \tilde{T}_n$$

$$S_n = \Lambda^{-1}(\tilde{S}_n)$$

end while

Simulating arrivals from a nonstationary Poisson process with cont. cum. rate $\Lambda(\cdot)$.

What is the density function of $(S_1, S_2, \ldots, S_n) \mid N(t) = n$ for a non-stationary Poisson process with rate function $\lambda(\cdot)$?

(Recall that this is the uniform order statistics for the stationary case)

$\boxed{\text{Thm. } 1.10}$ [CONDITIONAL DISTBN]

For any numbers s_j, $j = 1, 2, \dots, n$ satisfying $0 \le s_1 \le s_2 \cdots s_n \le t$

$$P\left(S_1 \le s_1, S_2 \le s_2, \dots, S_n \le s_n \mid N(t) = n\right)$$

$$= \frac{n!}{(\Lambda(t))^n} \int_0^{s_1} \cdots \int_{x_{n-2}}^{s_{n-1}} \int_{x_{n-1}}^{s_n} \lambda(x_n) \cdots \lambda(x_1)\, dx_n\, dx_{n-1} \cdots dx_1 ,$$

which is the distribution of the order statistics from a sample of n observations taken from $\lambda(\cdot)/\Lambda(t)$.

Heuristic

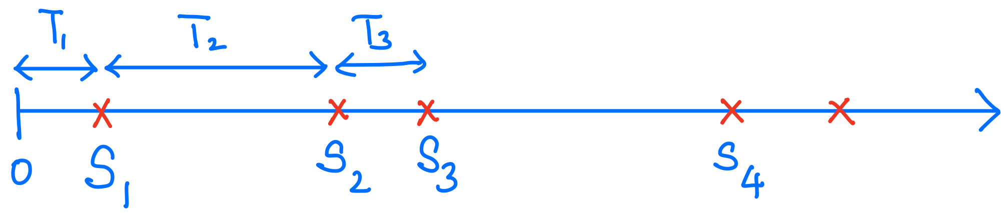
$$f_{S_1, S_2, \ldots S_n | N(t)}(s_1, s_2, \ldots s_n | n)\, ds_1\, ds_2 \ldots ds_n$$

$$= \frac{e^{-\Lambda(s_1)} \lambda(s_1)\, ds_1\; e^{-(\Lambda(s_2) - \Lambda(s_1))} \lambda(s_2)\, ds_2 \cdots e^{-(\Lambda(s_n) - \Lambda(s_{n-1}))} \lambda(s_n)\, ds_n\; e^{-(\Lambda(t) - \Lambda(s_n))}}{e^{-\Lambda(t)} (\Lambda(t))^n / n!}$$

$$= \frac{e^{-\Lambda(t)} \lambda(s_1) \lambda(s_2) \ldots \lambda(s_n)\, ds_1\, ds_2 \cdots ds_n}{e^{-\Lambda(t)} \Lambda(t)^n / n!}$$

$$= n!\, \frac{\lambda(s_1)}{\Lambda(t)} \frac{\lambda(s_2)}{\Lambda(t)} \cdots \frac{\lambda(s_n)}{\Lambda(t)}\, ds_1 \cdots ds_n$$

ALTERNATE CONSTRUCTIONS



CONDITION C_0

$$0 < S_1(\omega) < S_2(\omega) \cdots, \quad \sup_n S_n(\omega) = \infty$$

OR

$$T_1(\omega) > 0, \; T_2(\omega) > 0, \ldots \quad \sum_n T_n(\omega) = \infty$$

$\boxed{\text{CONDITION } C_1}$

$T_1, T_2, \ldots,$ are independent and identically distributed with parameter $\lambda > 0$.

$\boxed{\text{CONDITION } C_2}$

(i) For $0 < t_1 < t_2 \cdots < t_n$,

$X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$

are independent.

(ii) $X_t - X_s \sim \text{Poisson}\left(\lambda(t-s)\right)$.

$\boxed{\text{CONDITION } C_3}$

(i) For $0 < t_1 < t_2 \cdots < t_n$,

$X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$

are independent.

(ii) Distbn. of $X_t - X_s$ depends only on $t - s$.

Condition C_4

If $0 < t_1 < t_2 < \dots < t_k$ and
$\eta_j \geq 0$, $j = 1, 2, \dots, k$

(i) $P\left(X_{t_k+h} - X_{t_k} = 1 \,\middle|\, X_{t_j} = \eta_j, j \leq k \right)$

$$= \lambda h + o(h)$$

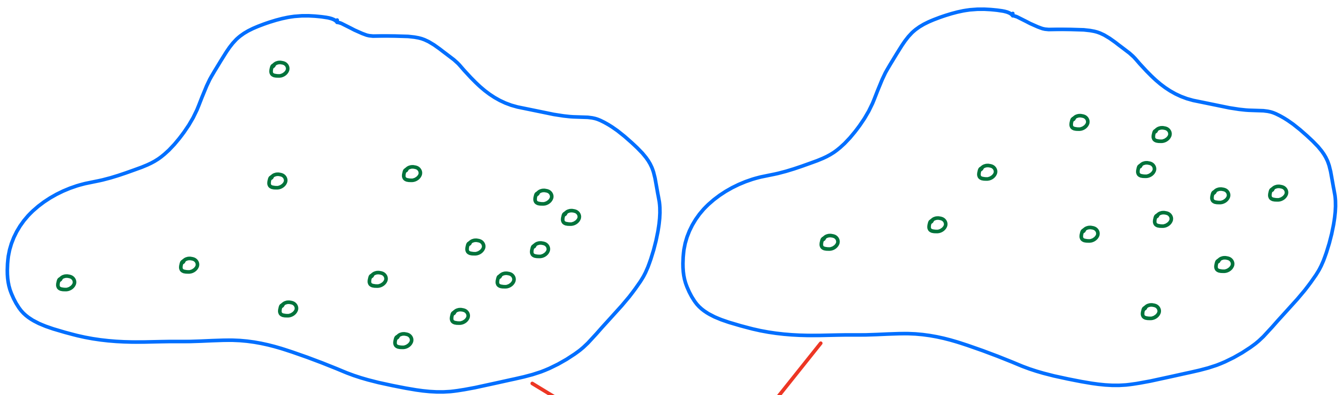
(ii) $P\left(X_{t_k+h} - X_{t_k} \geq 2 \,\middle|\, X_{t_j} = \eta_j, j \leq k \right)$

$$= o(h)$$

$\boxed{\text{Thm. 1.10}}$ [CONSTRUCTION EQUIV.]

Under CONDITION C_0, CONDITION C_1, CONDITION C_2, CONDITION C_3 and CONDITION C_4 are equivalent.

Point Processes



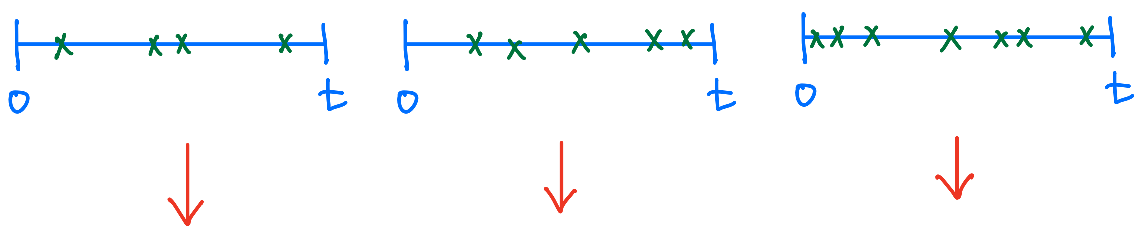
two realizations

The notion of point processes comes from viewing points in space as realizations of a probability law.

This is a powerful way of thinking about stochastic processes in general:

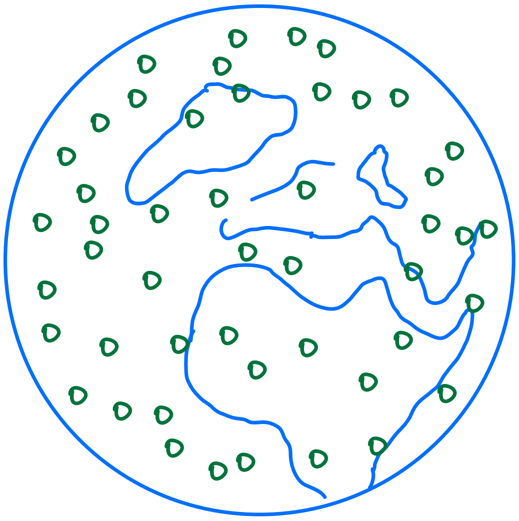
$$\omega \longmapsto X(\omega) \in S$$

appropriate space



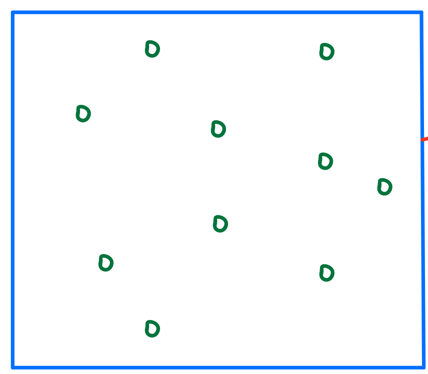
$X(\omega_1)$ $X(\omega_2)$ $X(\omega_3)$

POINT PROCESSES



Formally, a point process is a counting-measure in $\mathbb{R}^d$ valued random variable on $\mathbb{R}^d$.

EXAMPLE (Binomial Process)



$\rightarrow W \subset \mathbb{R}^2$

Let $X_1, X_2, \ldots, X_n \in \mathbb{R}^2$, n fixed, be uniformly distributed in $W \subset \mathbb{R}^2$ bounded.

The density of X_i is

$$f(x) = \begin{cases} 1/\mu_2(W), & x \in W \\ 0 & \text{o.w.}, \end{cases}$$

where $\mu_2(\cdot)$ denotes "area."

So, for any bounded set

$B \subset \mathbb{R}^2$,

$$P(X_i \in B) = \frac{\mu_2(B \cap W)}{\mu_2(W)}$$

$$N(B) := \sum_{j=1}^{n} \mathbb{I}\{X_j \in B\}$$

$$N(B) \sim \text{Binomial}\left(n, \frac{\mu_2(B \cap W)}{\mu_2(W)}\right)$$

Notice that $N(B_1)$ & $N(B_2)$ are not independent even if $B_1 \cap B_2 = \phi$ since

$$N(B_1) + N(B_2) \leq n.$$

The Poisson process can be viewed as a point process in $\mathbb{R}$, that is, as a counting-measure in $\mathbb{R}$ valued random variable.

Let's try to generalize to $\mathbb{R}^2$.

EXAMPLE (Spatial Poisson Proc.)

The spatial Poisson process, with rate $\lambda > 0$, is a point process in $\mathbb{R}^2$ such that

(PP1) for every bounded $B \subset \mathbb{R}^2$,

$$N(B) \sim \text{Poisson}\left(\mu_2(B)\right)$$

(PP2) B_j, $j = 1, 2, \ldots, n$ disjoint
$\Rightarrow N(B_j)$, $j = 1, 2, \ldots n$ are independent.

We can derive the spatial Poisson process through other sets of axioms, e.g.,

(i) $P(N(B) \geq 1) = \lambda \mu_2(B) + o(\mu_2(B))$

(ii) $P(N(B) \geq 2) = o(\mu_2(B))$

(iii) B_j, $j = 1, 2, \ldots, n$ disjoint
$\Rightarrow N(B_j)$, $j = 1, 2, \ldots n$ are independent.

$\boxed{\text{Thm. 1.11}}$ Let N be a Poisson point process in $\mathbb{R}^2$ with rate $\lambda > 0$. Let $W \subset \mathbb{R}^2$ be such that $0 < \mu_2(W) < \infty$. Then,

$$P\Big(N(B) = k \,\Big|\, N(W) = n\Big)$$

$$= \binom{n}{k}\left(\frac{\mu_2(B)}{\mu_2(W)}\right)^{k}\left(1 - \frac{\mu_2(B)}{\mu_2(W)}\right)^{n-k},$$

$k \leq n$, $B \subseteq W$.

Furthermore,

$$N(B_1), N(B_2), \ldots, N(B_m) \,\Big|\, N(W) = n$$

$$B_j \subseteq W$$

is the same as the joint distribution of these variables in a binomial process.

We can generalize further!

Let S be a space and Λ a measure on it. (Strictly, S is a locally compact metric space, and Λ a measure which is finite on every compact set and which has no atoms.)

The Poisson process on S with "cumulative rate" Λ is a point process on S such that

(PP1) for every compact $\mathcal{B} \subset S$,

$$N(\mathcal{B}) \sim \text{Poisson}(\Lambda(\mathcal{B}))$$

(PP2) $\mathcal{B}_j$, $j = 1, 2, \ldots, n$ disjoint and compact $\Longrightarrow$ $N(\mathcal{B}_j)$, $j = 1, 2, \ldots n$ are independent.

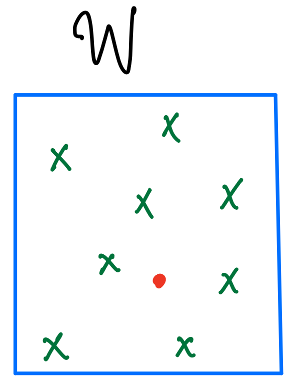
$\boxed{\text{Thm. 1.12}}$ Let N be a Poisson point process in $\mathbb{R}^2$ with "cumulative rate" Λ. Let $W \subset \mathbb{R}^2$ be such that $0 < \Lambda(W) < \infty$. Then,

$$P\left(N(B) = k \mid N(W) = n\right)$$

$$= \binom{n}{k} \left(\frac{\Lambda(B)}{\Lambda(W)}\right)^k \left(1 - \frac{\Lambda(B)}{\Lambda(W)}\right)^{n-k},$$

$k \le n$, $B \subseteq W$.

EXAMPLE



Cabs are available in a finite city $W \subset \mathbb{R}^2$ according to a Poisson point process with cumulate rate Λ.

The next demand loc. in W happens according to the density $f(x)$, $x \in W$. Find the expected shortest distance to a cab.