Recall one of the constructions of of the Poisson process.

CONDITION C_4

If $0 < t_1 < t_2 < \dots < t_k$ and $n_j \geq 0$, $j = 1, 2, \dots, k$

(i) $$P\left(X_{t_k+h} - X_{t_k} = 1 \,\middle|\, X_{t_j} = n_j,\ j \leq k \right) = \lambda h + o(h)$$

(ii) $$P\left(X_{t_k+h} - X_{t_k} \geq 2 \,\middle|\, X_{t_j} = n_j,\ j \leq k \right) = o(h)$$

Observe

- (i) and (ii) imply a sense of future state distribution not depending on the past

- X_t is non-decreasing and $X_t \in \mathbb{Z}^+$.

Can we generalize?

Markov Chain

Let E be a countable set. The stochastic process $\{Y_t, \ t \geq 0\}$ is said to be a Markov process with state space E provided that for any $t, s \geq 0$, and $j \in E$

$$P\left(Y_{t+s} = j \mid Y_u ; u \leq t\right)$$
$$= P\left(Y_{t+s} = j \mid Y_t\right) \qquad \text{—— (1)}$$

Observe

- RHS of (1) can depend on s, t or j, Y_t.

- When

"transition function"

$$P(Y_{t+s} = j \mid Y_t = i) = P_s(i,j)$$

is independent of t, we say that the Markov chain is time-homogeneous.

- continuous time versus discrete time Markov chain.

Discrete time Markov chain

"discrete time"

$\{X_n, n \in \mathbb{N}\}$ is called a

discrete time Markov chain if

$$P(X_{n+1} = j \mid X_0, X_1, \dots, X_n)$$

$$= P(X_{n+1} = j \mid X_n).$$

for all $j \in E$ and $n \in \mathbb{N}$.

state space E is a countable set.

For convenience,

$$P(X_{n+1} = j \mid X_n = i) = P(i,j),$$

"transition matrix"

that is, there is no dependence on n.

Notice:

(i) $0 \leq P(i,j) \leq 1, \quad i,j \in E.$

(ii) $\displaystyle\sum_{j \in E} P(i,j) = 1 \quad \forall i$

Matrices that satisfy (i) and (ii) are called Markov matrices.

$\boxed{\text{Theorem 3.1}}$ (only current state matters)

For $n, m \in \mathbb{N}$, and $i_0, i_1, \dots i_m \in E$

$$P\left(X_{n+m} = i_m, \dots, X_{n+1} = i_1 \mid X_n = i_0, X_{n-1} = i_{-1}, \dots, X_0 = i_{-n}\right)$$

$$= P(i_0, i_1) P(i_1, i_2) \cdots P(i_{m-1}, i_m).$$

$$= P\left(X_{n+m} = i_m, \dots, X_{n+1} = i_1 \mid X_n = i_0\right)$$

If $\pi(\cdot)$ is a distribution on E, and $P(X_0 = i) = \pi(i)$, $i \in E$,

$$P\left(X_0 = i_0, \dots, X_m = i_m\right) = \sum_{j \in E} \pi(j) \prod_{j=0}^{m-1} P(i_j, i_{j+1}).$$

$\square$

- "path" to $X_n = i_0$ does not matter

  Markov

- current time n does not matter

  time homogeneity

Notice, therefore

$$P(X_{n+2} = k \mid X_n = i)$$

$$= \sum_{j \in E} P(i,j) P(j,k)$$

$$P(X_{n+3} = k \mid X_n = i) \qquad P(X_{n+3} = k \mid X_{n+1} = h)$$

$$= \sum_{h \in E} P(i,h) \left(\sum_{j \in E} P(h,j) P(j,k) \right)$$

And,

$$P(X_{m+n} = j \mid X_0 = i)$$

$$= \sum_{h \in E} P(X_m = h \mid X_0 = i) \times P(X_{m+n} = j \mid X_m = h)$$

$$\Rightarrow \boxed{P^{(m+n)} = P^{(n)} P^{(m)} \quad m, n \in \mathbb{N}}$$

Notation: $P^{(t)}_{(i,j)} = P(X_t = j \mid X_0 = i)$

$\boxed{\text{Theorem 3.2}}$

For any $m \in \mathbb{N}$,

$$P(X_{n+m} = j \mid X_n = i) = P^m(i,j) \,,$$

$i, j \in E; \; n \in \mathbb{N}.$

(i,j)-th element of P^m.

$\square$

In open form, for any $m, n \in \mathbb{N}$

$$\boxed{P^{(n+m)}(i,j) = \sum_{h \in E} P^{m}(i,h)\, P^{n}(h,j) \qquad i,j \in E;\ h \in E.}$$

Chapman-Kolmogorov Equations

EXAMPLE

Suppose $Y_j, j \geq 1$ are iid discrete valued random variables with distribution π_k, $k = 0, 1, 2, \ldots$

Suppose

$$X_n := \begin{cases} 0 & n = 0 \\ \sum_{j=1}^{n} Y_j, & n \geq 1. \end{cases}$$

Then $\{X_n, n \in \mathbb{N}\}$ is a Markov chain with

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & \cdots & \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ \\ \end{array} & \begin{bmatrix} p_0 & p_1 & p_2 & \cdots & & \\ & p_0 & p_1 & p_2 & \cdots & \\ & 0 & p_0 & p_1 & \cdots & \\ & & & p_0 & \cdots & \\ & & & & \ddots & \end{bmatrix} \end{array}$$

EXAMPLE

When an equipment fails, it is immediately replaced by another. Let p_k denote the time equipment lasts for k units of time.

X_n := "remaining life" at time n.

(Assume lifetimes Z_n are iid.)

Since,

$$X_{n+1}(\omega) = \begin{cases} X_n(\omega) - 1 & X_n(\omega) \geq 1 \\ Z_{n+1}(\omega) - 1 & \text{o.w.} \end{cases}$$

Therefore $\{X_n,\ n \geq 1\}$ is a Markov chain.

$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \\ \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & & & \end{array} \\
\begin{bmatrix}
p_1 & p_2 & p_3 & \cdot & \cdot & \cdot \\
1 & 0 & 0 & \cdot & \cdot & \cdot \\
0 & 1 & 0 & \cdot & \cdot & \cdot \\
 & & & \ddots & &
\end{bmatrix}
\end{array}
$$

Theorem 3.3

Fix $n \in \mathbb{N}$, and let Y be a bounded function of the random variables $X_n, X_{n+1}, \dots$

Then

$$\mathbb{E}\left[Y \mid X_0, \dots X_n\right] = \mathbb{E}\left[Y \mid X_n\right]$$

Proof ?

Theorem 3.4

Let f be a bounded function on $E \times E \times \cdots$ and let

$$g(i) = \mathbb{E}\left[f(X_0, X_1, \ldots) \mid X_0 = i\right]$$

Then, for any $n \in \mathbb{N}$,

$$\mathbb{E}\left[f(X_n, X_{n+1}, \ldots) \mid X_0, \ldots, X_n\right] = g(X_n).$$

Stopping Time.

The r.v. T is said to be a stopping time if the occurrence or non-occurrence of the event $T \leq n$ can be determined by looking at $X_0, X_1, \ldots X_n$.

EXAMPLES

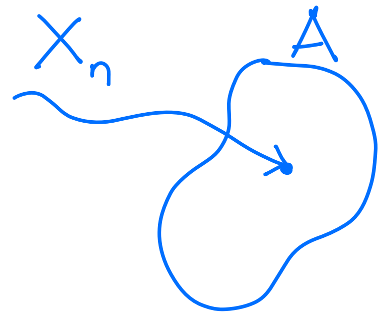
(i) first time X visits a certain state ✓

(ii) first time X visits a certain set of states ✓

(iii) k-th time X visits a fixed set of states ✓

(iv) the last time a state is visited ✗

Hitting Time



$$T_A := \inf\{n > 0 : X_n \in A\}$$

"earliest positive time the chain is in A"

We will abuse notation and write T_x instead of $T_{\{x\}}$.

$$\rho_{xy} := P\left(T_y < \infty \mid X_0 = x\right)$$

"probability of ever hitting y from x"

$$\rho_{xy} = \sum_{n=1}^{\infty} P\left(T_y = n \mid X_0 = x\right)$$

Recurrent State

A state y such that

$$\rho_{yy} = 1.$$

Transient State

A state y such that

$$\rho_{yy} < 1.$$

Denote

$$\mathbb{I}_y(x) := \begin{cases} 1 & x = y \\ 0 & \text{o.w.} \end{cases}$$

$$N(y) = \sum_{n=0}^{\infty} \mathbb{I}_y(X_n)$$

"number of visits to y"

Notice

$$P\left(N(y) \geqslant 1 \mid X_0 = x\right) = f_{xy}$$

$$P\left(N(y) \geqslant 2 \mid X_0 = x\right) = f_{xy} f_{yy}$$

$$P\left(N(y) \geqslant m \mid X_0 = x\right) = f_{xy} f_{yy}^{m-1}$$

$m \geqslant 1$.

Proof?

Expected Number of Visits

$$G(x,y) := \mathbb{E}\left[N(y) \,\middle|\, X_0 = x \right]$$

$$= \mathbb{E}\left[\sum_{n=0}^{\infty} \mathbb{I}_y(X_n) \,\middle|\, X_0 = x \right]$$

$$= \sum_{n=0}^{\infty} P^n(x, y)$$

Theorem 3.5

Let y be a transient state.

Then

$$P(N(y) < \infty \mid X_0 = x) = 1,$$

and

$$G(x,y) = \frac{\rho_{xy}}{1-\rho_{yy}}, \quad x \in S.$$

Let y be a recurrent state.

Then

$$P(N(y) < \infty \mid X_0 = x) = 1 - \rho_{xy}$$

and

$$G(x,y) = \begin{cases} 0 & \rho_{xy} = 0 \\ \infty & \rho_{xy} > 0. \end{cases}$$

$\square$

A Markov chain is called a transient chain if all its states are transient and a recurrent chain if all its states are recurrent.

A Markov chain with a finite state space cannot be transient. Why?

We say that $\underline{x \text{ leads to } y}$

if $\underline{p_{xy} > 0}$.

$$\text{"}x \text{ leads to } y\text{"} \equiv \text{"}x \to y\text{"}$$

Recall that ρ_{xy} can be expressed in different ways

$$\rho_{xy} = \sum_{n=1}^{\infty} P(T_y = n \mid X_0 = x)$$

$$= \lim_{n} P(T_y < n \mid X_0 = x)$$

$$= P(N(y) \geq 1 \mid X_0 = x) \qquad \text{—— (1)}$$

Examining (1), we see that x leads to y if and only if

$$P^n(x, y) > 0$$

for some $n \geq 1$.

By the same logic, if $x \to y$ and $y \to z$, then $x \to z$.

$\boxed{\text{Theorem 3.6}}$

If x is a recurrent state, and $x \to y$, then

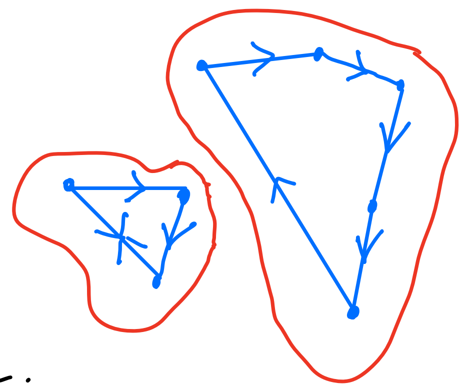
(i) y is a recurrent state;

(ii) $\rho_{xy} = \rho_{yx} = 1$.

$\blacksquare$

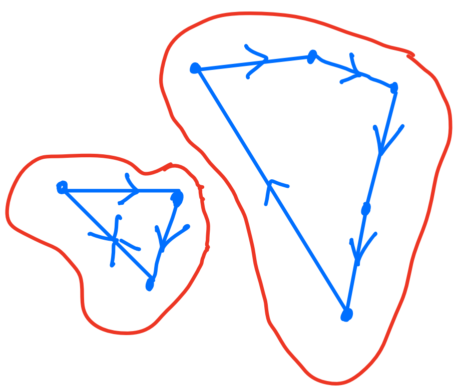
Proof?

A set of states C is said to be closed if no state in C leads to one outside.



$$p_{xy} = 0 \qquad x \in C,\ y \notin C$$

A closed set C is called <u>irreducible</u> if for any pair $x, y \in C$, we have that x leads to y.



An irreducible chain is a chain whose state space is irreducible.

$\boxed{\text{Theorem 3.7}}$

Let C be an irreducible closed set of recurrent states.

Then $\rho_{xy} = 1$, $P(N(y) = \infty \mid X_0 = x) = 1$ and $G(x, y) = \infty$ for all $x, y \in C$.

$\blacksquare$

Proof ?

Theorem 3.8

Let C be a finite irreducible closed set of states. Then, every state in C is recurrent.

$\boxtimes$

Proof?

In fact every irreducible set has all transient states or all recurrent states.

Why?

Example

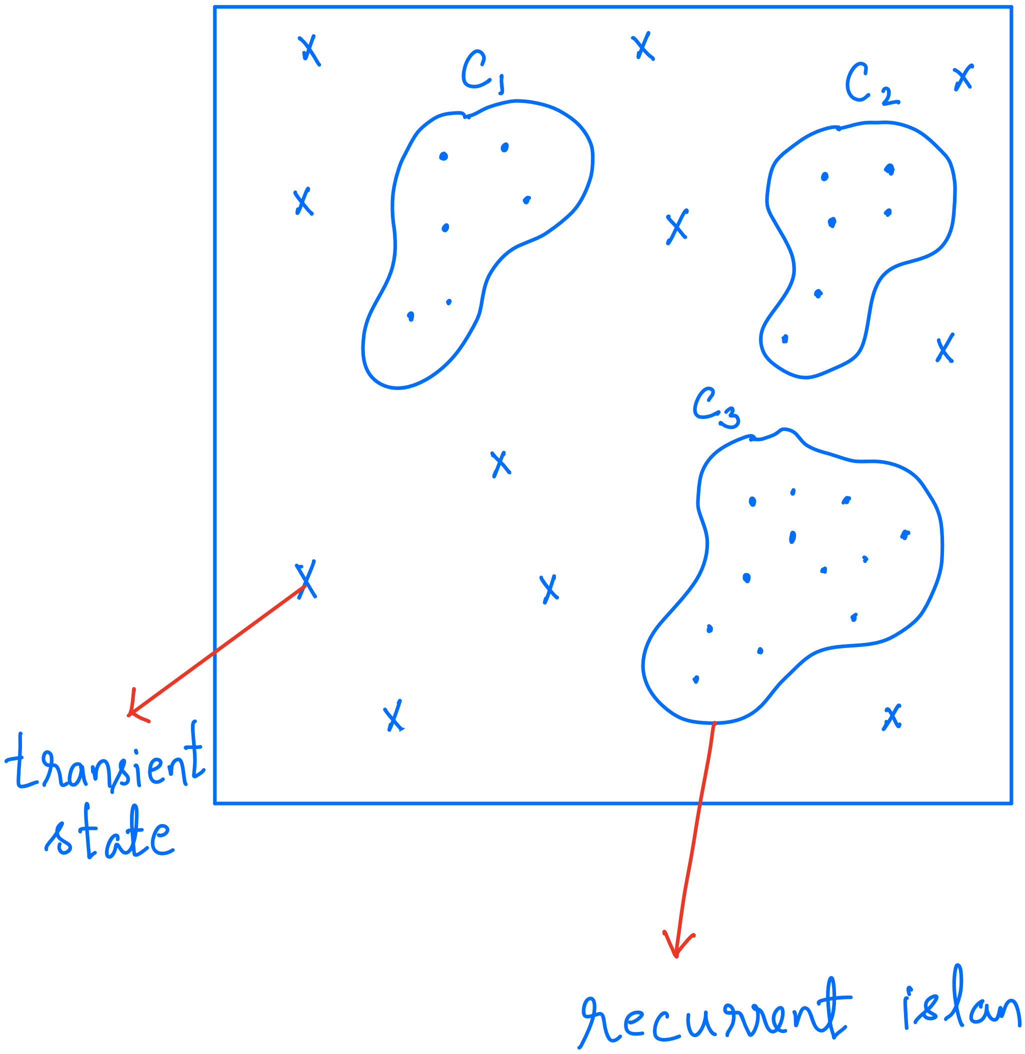
$$
\begin{array}{c|c}
 & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1/4 & 1/2 & 1/4 & 0 & 0 & 0 \\
0 & 1/5 & 2/5 & 1/5 & 0 & 1/5 \\
0 & 0 & 0 & 1/6 & 1/3 & 1/2 \\
0 & 0 & 0 & 1/2 & 0 & 1/2 \\
0 & 0 & 0 & 1/4 & 0 & 3/4
\end{bmatrix}
\end{array}
$$

Classify the states.

$\boxed{\text{Theorem 3.9}}$ (Decomposition)

Suppose the set S_R of recurrent states is nonempty. Then, S_R is the union of a finite or countably infinite number of disjoint irreducible closed sets $C_1, C_2, \ldots$

$\square$



MCs enter a recurrent island and remain there forever, visiting each state i.o., or remain in S_T, never entering any of the islands.

If C is a closed irreducible set of recurrent states and x is a transient state, we can calculate $\rho_C(x)$ by solving a linear system.

$$\rho_C(x) = \sum_{y \in C} P(x,y) + \sum_{y \in S_T} P(x,y)\,\rho_C(y)$$

$$x \in S_T$$

$|S_T|$ equations, $|S_T|$ variables

Martingales

Suppose that the sequence $X_n, n \geq 0$ satisfies

$$\mathbb{E}\left[X_{n+1} \mid X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n\right]$$
$$= x_n.$$

Such a sequence is called a martingale.

Notice that if $\{X_n, n \geq 0\}$ is a martingale, then the mean remains fixed.

$$\mathbb{E}[X_0] = \mathbb{E}[X_1] = \cdots$$

Let $\{X_n, n \geq 1\}$ be a Markov chain with state space $\{0, 1, 2, \ldots, d\}$, and transition probability matrix P such that

$$\sum_{y=0}^{d} y\, P(x,y) = x, \quad x = 0, 1, \ldots, d$$

The $\{X_n, n \geq 1\}$ is a martingale with 0 and d being absorbing states. Why?

Also, if 0 and d are the only absorbing states, then states $1, 2, \ldots, d-1$ are necessarily transient. Why?

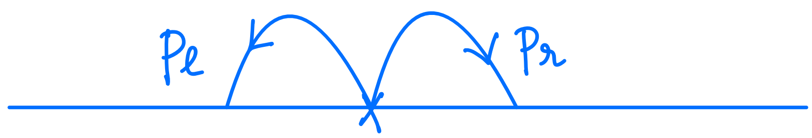
Furthermore,

$$p_{xd} = \frac{x}{d}, \quad p_{x0} = 1 - \frac{x}{d}$$

$\blacksquare$

Proof?

Birth and Death Chain



A Birth and Death chain is a MC supported on $\{0, 1, 2, \dots\}$ with the transition function

$$P(x,y) = \begin{cases} p_x & y = x+1 \\ r_x = 1 - p_x - q_x & y = x \\ q_x & y = x-1 \end{cases}$$

$q_0 = 0$; $p_x > 0$ & $q_x > 0$ for $x > 0$.

Main Question: $P(T_a < T_b)$?

$$u(x) := P(T_a < T_b \mid X_0 = x)$$

$$u(a) = 1, \quad u(b) = 0$$

$$u(x) = p_x u(x+1) + r_x u(x) + q_x u(x-1)$$

$$a < x < b$$

$$\Rightarrow q_x\big(u(x) - u(x-1)\big) = p_x\big(u(x+1) - u(x)\big)$$

$$u(x+1) - u(x) = \frac{q_x}{p_x}\big(u(x) - u(x-1)\big)$$

$$a < x < b$$

Recurse backward and use boundary conditions to solve:

$$u(x) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y} \qquad a < x < b$$

$$\gamma_y := \prod_{j=1}^{y} (q_j / p_j) \qquad y \geq 1$$

$$\gamma_0 := 1$$

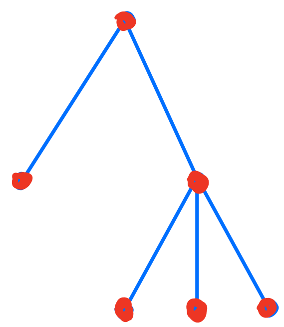
Branching Chain

X_n := number of "particles" at time n, $n \geq 0$.

Each particle at time n gives rise to ξ "progeny" independently, where ξ has pmf f.

What is the probability of "extinction": $P(T_0 < \infty \mid X_0 = x_0)$?

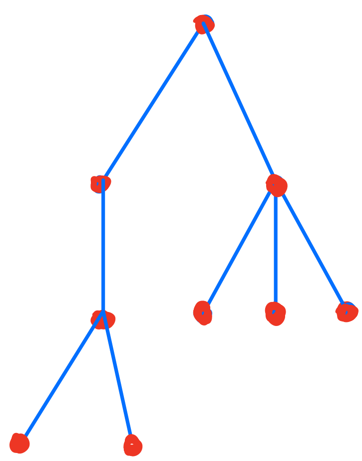
Understand the Model:



$X_0 = 1$

$X_1 = 2$

$X_2 = 3$



$X_0 = 1$

$X_1 = 2$

$X_2 = 4$

$X_3 = 2$