Let $P(x,y)$, $x,y \in S$ be a transition prob. matrix.

Let $\pi(x)$, $x \in S$ be a

such that $\sum_x \pi(x) = 1$, $\pi(x) \geq 0$ and

$$\boxed{\sum_x \pi(x) P(x,y) = \pi(y)}$$

$\left(\pi = \pi P, \ \pi \text{ is a row vector}\right)$

Then, $\pi(x)$, $x \in S$ is called a <u>stationary</u> <u>distribution</u>.

eigen interpret

A stationary distbn need not exist, nor it need be unique.

For example,

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & \cdots \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & \cdots & \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & & \end{bmatrix} \end{array} \rightarrow$$

$P(x, x+1) = 1$ no stationary distribn.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow$$

$\pi = [0, 1]$, $\pi = [1, 0]$ are both stationary.

Notice that if $\pi(x), x \in S$ is stationary then,

$$\sum_x \pi(x) P^n(x,y) = \pi(y).$$

How?

Interpret above as:

- $X_0 \sim \pi \implies X_n \sim \pi$
- if π is stationary w.r.t P, then it is stationary w.r.t P^n, $n \geq 1$

Suppose, conversely, that X_n's distbn is independent of n. Then, $\forall\, y \in S$

$$P(X_0 = y) = P(X_1 = y)$$
$$= \sum_x P(X_0 = x)\, P(x, y),$$

and thus X_0's distbn is stationary.

Now suppose P satisfies

$$\lim_{n\to\infty} P^n(x, y) = \pi(y), \qquad \text{———(1)}$$

then

$$\boxed{\lim_{n\to\infty} P(X_n = y) = \pi(y)} \qquad \text{———(2)}$$

It is important that π in (2) need not be stationary.

(2) is a statement about the marginal limiting behavior.

Proof of (2)?

If (2) is satisfied, and $\pi(x), x \in S$ is stationary, then $\pi(\cdot)$ is a unique stationary distbn. Why?

EXAMPLE

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \end{array}$$

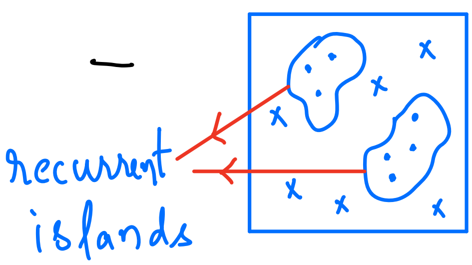
Suppose $0 < p < 1$, $0 < q < 1$.

$$P^n = \begin{bmatrix} \frac{q}{p+q} + (1-p-q)^n \frac{p}{p+q} & \frac{p}{p+q} - (1-p-q)^n \frac{p}{p+q} \\ \frac{q}{p+q} - (1-p-q)^n \frac{q}{p+q} & \frac{p}{p+q} + (1-p-q)^n \frac{q}{p+q} \end{bmatrix}$$

$$\pi(0) = \frac{q}{p+q} \qquad \pi(1) = \frac{p}{p+q}$$

In general, (2) and π being stationary are needed for uniqueness of π.

— transient chain satisfies (1) but the π is not stationary.

— 

(1) is not satisfied but a stationary distbn exists.

What fraction of time (in the long term) does a MC spend in a particular state?

First, note that

$$\lim_{n \to \infty} P^n(x, y)$$

is not a good measure to answer the previous question (especially if y is recurrent). Why?

Better ways to measure:

$$\lim_{n} \frac{N_n(y)}{n}$$

$$\lim_{n} \frac{G_n(x,y)}{n}$$

Where

$$N_n(y) := \sum_{m=0}^{n} \mathbb{I}_y(X_m)$$

$$G_n(x,y) := \mathbb{E}\left[N_n(y) \mid X_0 = x\right]$$

"total number of visits to y"

$\boxed{\text{Theorem 4.1a}}$

Suppose y is transient.

Then,

$$\lim_n N_n(y) < \infty \quad \underbrace{\text{w.p. }1}_{\text{"with prob. one"}}$$

and hence,

$$\lim_n \frac{N_n(y)}{n} = 0 \quad \text{w.p. }1.$$

And,

$$\lim_n \frac{G_n(x,y)}{n} = 0.$$

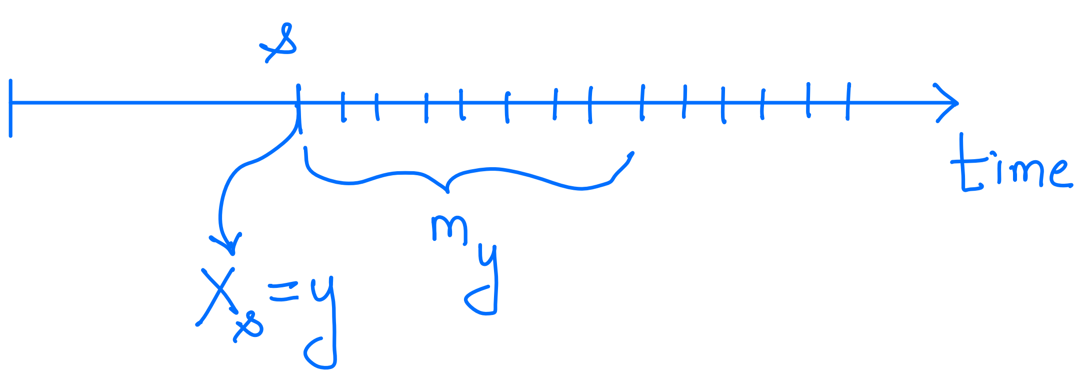
Define

$$m_y := \mathbb{E}\left[T_y \mid X_0 = y \right]$$

"average return time"

$T_y = \min\{n \geq 1 : X_n = y\}$

$\left(m_y = \infty \text{ is allowed} \right)$



Theorem 4.1b

Suppose y is recurrent.

Then,

$$\lim_{n} \frac{N_n(y)}{n} = \frac{\mathbb{I}(T_y < \infty)}{m_y} \quad \text{w.p.1.}$$

$$\lim_{n} \frac{G_n(x,y)}{n} = \frac{\rho_{xy}}{m_y}.$$

Proof?

$\boxtimes$

Null Recurrence, Positive Recurrence

A recurrent state y is said to be null recurrent if $m_y := \mathbb{E}[T_y \mid X_0 = y] = \infty$.

The state y is positive recurrent if $m_y < \infty$.

It follows from Theorem 4.1b that y is positive recurrent or null recurrent according to whether $\lim_{n\to\infty} \frac{G_n(y,y)}{n}$ is >0 or 0.

(In fact $\lim_n P^n(y,y) = 0$ if y is null recurrent.)

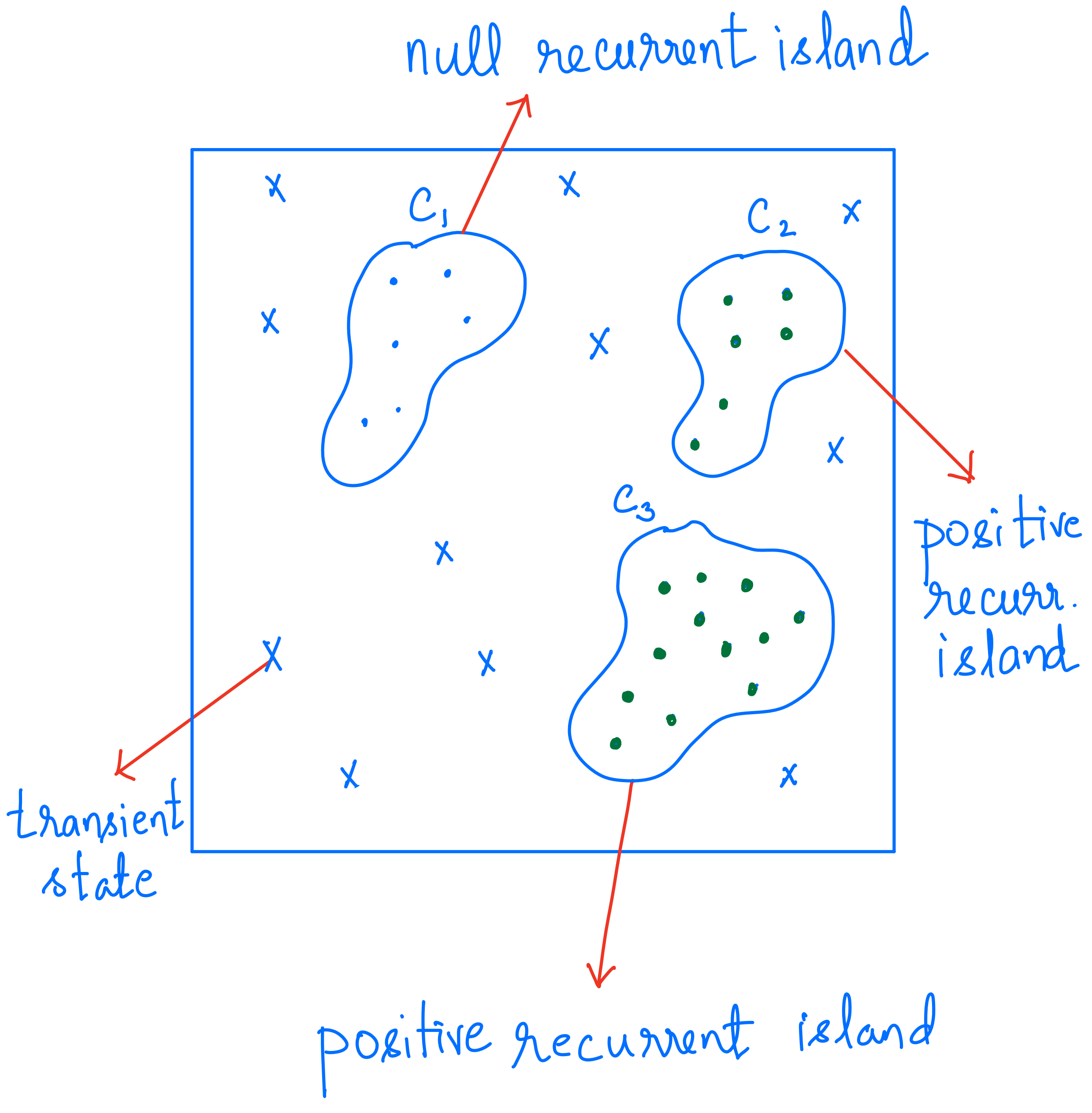
$\boxed{\text{Theorem 4.2}}$

Let x be a positive recurrent state, and suppose $x \longrightarrow y$. Then, y is positive recurrent.

(Proof?)

From Thm. 4.2, it is clear that if x is null recurrent and if $x \to y$ then y is null recurrent as well.

Enhanced Depiction



Can anything further be said for a closed finite set?

Lets see . . .

Suppose C is closed and $|C| < \infty$. The following further statements are true.

A. There exists at least one positive recurrent state.

B. There exist no null recurr. states.

C. If C is irreducible as well, then every state is positive recurrent.

(Proof?)

Suppose π is a stationary distribution. Then $\pi(x) = 0$ if x is null recurrent or transient.

$\boxed{\text{Remark}}$

Suppose $\{a_n(x) \quad n \geq 1\}$ is a positive-valued sequence of functions (with domain $D \subset \mathbb{R}$) such that for each x,

$$a_n(x) \longrightarrow a(x) \quad \text{as} \quad n \to \infty.$$

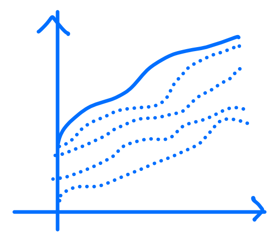
Under what conditions can we say:

$$\int a_n(x)\, dx \to \int a(x) \,?$$

Two Key Conditions

(i) (MCT) If $a_n \uparrow a$, then

$$\int a_n \longrightarrow \int a .$$



(limit can be ∞)

(ii) (DCT) If $a_n \leq g$ for all n, and $\int g < \infty$, then

$$\int a_n \longrightarrow \int a .$$

Theorem 4.3

An irreducible positive recurrent Markov chain has a unique stationary distbn. π, given by

$$\pi(x) = \frac{1}{m_x}, \quad x \in S.$$

$\blacksquare$

(Proof?)

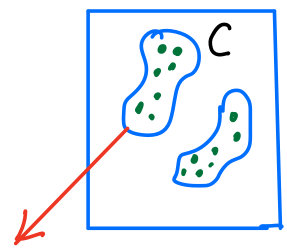
The following hold.

D. An irreducible Markov chain is positive recurrent if and only if it has a stationary distbn.

E. If a Markov chain with $|S| < \infty$ is irreducible, it has a unique stationary distbn.

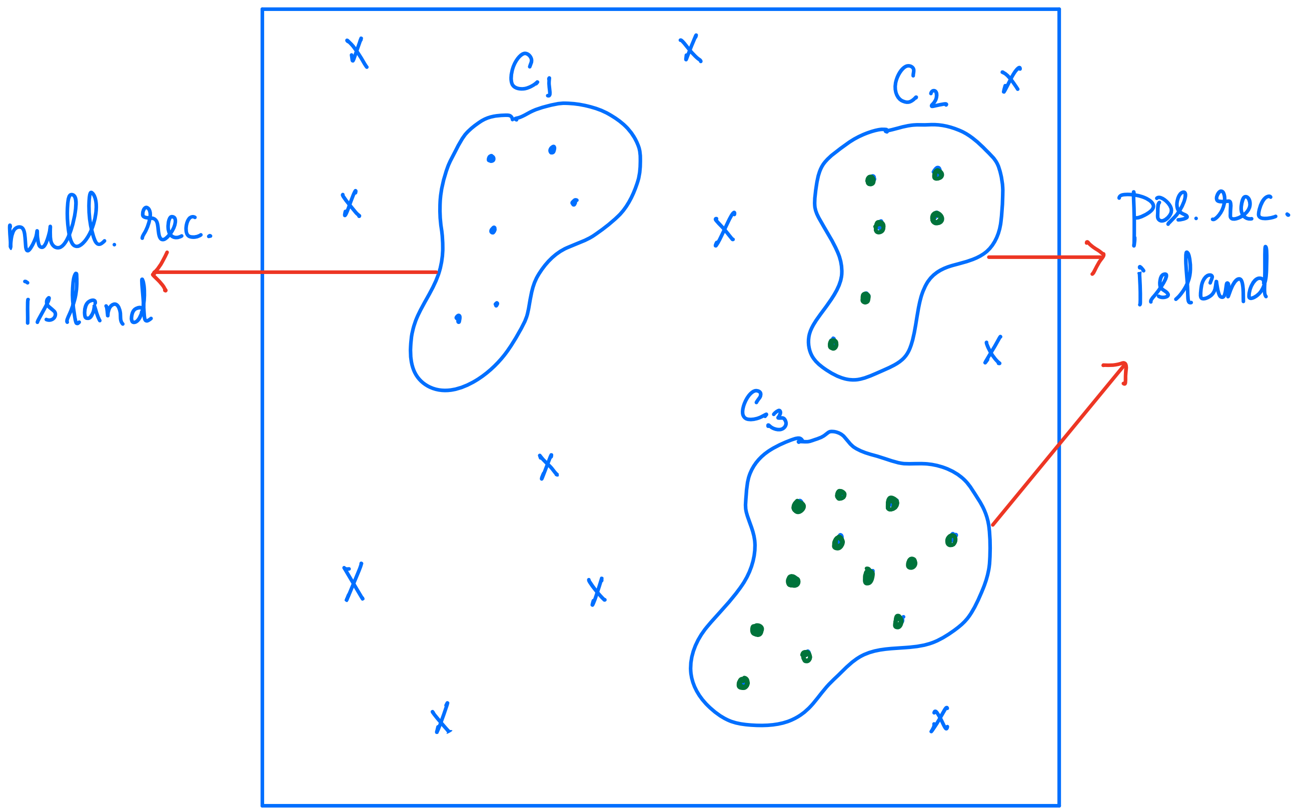
F. For an irreducible positive recurrent MC, $\lim_{n} \frac{N_n(x)}{n} = \pi(x)$.

We see from Theorem 4.3 that if C is an irreducible closed set of positive recurrent states. Then



$$\pi(x) = \begin{cases} \dfrac{1}{m_x} & x \in C \\ 0 & x \notin C \end{cases}$$

is a stationary distribution.



$$\pi_2 = \begin{cases} 1/m_x & x \in C_2 \\ 0 & \text{o.w.} \end{cases}$$

Let :

$$\pi_3 = \begin{cases} 1/m_x & x \in C_3 \\ 0 & \text{o.w} \end{cases}$$

The $\alpha \pi_2 + (1-\alpha)\pi_3$ is stationary for any $\alpha \in [0,1]$.

Our discussion started with wanting a measure of long-term behavior.

Two Possibilites:

(i) $\lim_{n} \frac{G_n(x,y)}{n}$ ✓

(ii) $\lim_{n} P^n(x,y)$ ✗

We have shown that for an irreducible +ve recurrent chain,

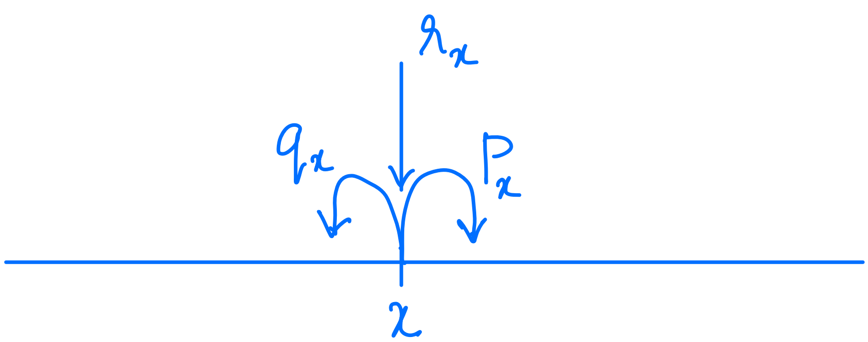
$$\lim_{n} \frac{G_n(x,y)}{n} = \frac{1}{m_y} = \pi(y)$$

Under what conditions is

$$\lim_{n} P^n(x,y) = \lim_{n} \frac{G_n(x,y)}{n} \;?$$

Ans: Aperiodicity

Recall the birth death process on the integers.



Suppose $r_x = 0 \ \forall x$.

Then, clearly, $P^{2n+1}(x,x) = 0 \ \forall n \geq 1$.

How to formally characterize this behavior?

For a state x, define

$$d_x := \gcd\left\{ n \geq 1 : P^n(x,x) > 0 \right\}$$

"period of x"

"greatest common divisor"

$d_x \geq 1$ by definition

$d_x \geq 2$ in the previous example.

The following theorem states communicating states have the same period.

$\boxed{\text{Theorem 4.4}}$

If $x \leftrightarrow y$, then $d_x = d_y$.

$\blacksquare$

(Proof?)

An irreducible chain thus has a common period $d = d_x \; \forall x$

An irreducible chain is called apeiodic if $d = 1$.

Theorem 4.5

An irreducible aperiodic positive recurrent chain satisfies

$$\lim_{n\to\infty} P^n(x,y) = \pi(y) \quad \forall x,y$$

where π is the unique stat. distbn.

If $d>1$, then there exists $r=r(x,y)<d$, such that

$$\lim_{m\to\infty} P^{md+r}(x,y) = d\pi(y). \quad \forall x,y$$

and $P^n(x,y)=0$ if $n\neq md+r$.

Interpretation:

You "observe" the system on a different time scale.