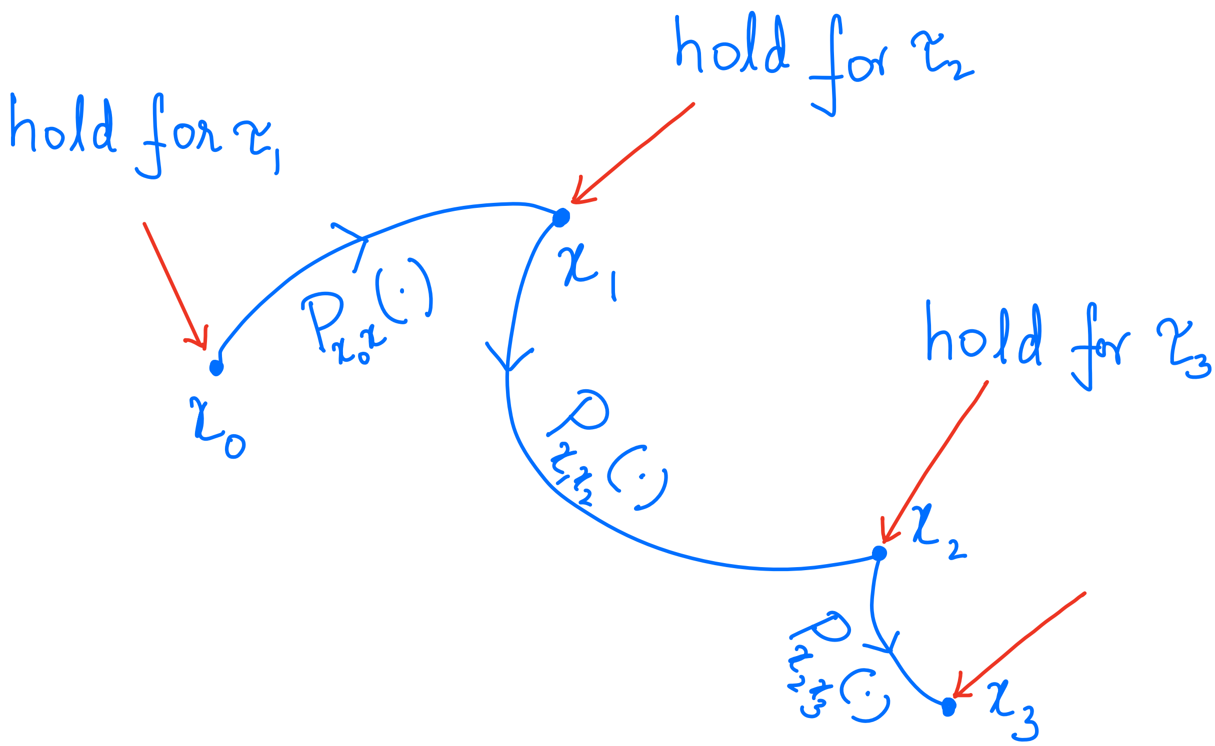
Jump Process



$$X_t = \begin{cases} x_0 & 0 \le t \le \tau_1 \\ x_1 & \tau_1 \le t \le \tau_1 + \tau_2 \\ \vdots & \\ x_n & \end{cases}$$

Associated with a non-absorbing $x \in S$

(i) residence time R_x with distbn F_x

(ii) jump state J_x with probs. Q_{xy}, $x, y \in S$, with

$$Q_{xx} = 0; \quad \sum_y Q_{xy} = 1$$

The r.v.s J_x and R_x are independent, so that

$$P(R_x \leq r, J_x = y \mid X_0 = x) = F_x(r)\, Q_{xy}.$$

If $x \in S$ is absorbing,

$$Q_{xy} = \delta_{xy} = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}$$

$X(t)$ is the state at time t, and

$$P_{xy}(t) := P\big(X(t)=y \mid X(0)=x\big)$$

Clearly,

$$\sum_y P_{xy}(t) = 1 \quad \forall x \in S.$$

Markov Pure Jump:

For $s_1 \le s_2 \le \cdots \le s_n \le s \le t$, and each $x_1, x_2, \ldots, x_n, x, y \in S$:

$$P\Big(X(t) = y \,\Big|\, X(s_1) = x_1, \ldots, X(s_n) = x_n, X(s) = x\Big)$$
$$= P\big(X(t) = y \,\big|\, X(s) = x\big).$$
$$= P_{xy}(t-s)$$

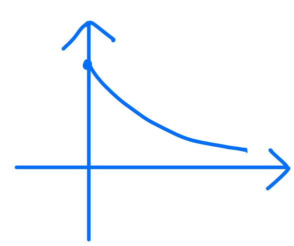
"Markov + memoryless"

The Markov property holds iff R_x is exponential for each non-absorbing state x.

Recall: $R_x \sim \exp(q_x)$ implies

$$F_x(t) = 1 - e^{-q_x t}, \quad t \geq 0$$

$$f_x(t) = q_x e^{-q_x t}, \quad t \geq 0$$



$$\mathbb{E}[R_x] = 1/q_x$$

Observe:

$$P_{xy}(t+s) = \sum_z P_{xz}(t)\, P_{zy}(s)$$

$s \geq 0;\ t \geq 0$

Chapman Kolmogorov Eqn

$$P_{xy}(t) =$$

$$\delta_{xy} e^{-q_x t} + \int_0^t q_x e^{-q_x s} \sum_{z \neq x} Q_{xz} P_{zy}(t-s)\, ds$$

$t \geq 0$

(How?)

Change of variable gives,

$$P_{xy}(t) = \delta_{xy} e^{-q_x t} +$$

$$q_x e^{-q_x t} \int_0^t e^{+q_x s} \sum_{z \neq x} Q_{xz} P_{zy}(s) \, ds$$

$$t \geqslant 0$$

And,

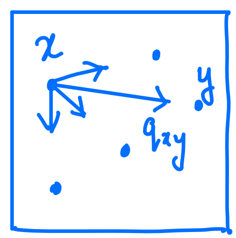
$$P'_{xy}(t) = -q_x P_{xy}(t) + q_x \sum_{z \neq x} Q_{xz} P_{zy}(t)$$

$$t \geqslant 0$$

$$P'_{xy}(0) = -q_x \delta_{xy} + q_x \sum_{z \neq x} Q_{xz} \delta_{zy}$$

$$= -q_x \delta_{xy} + q_x Q_{xy}, \quad x, y \in S$$

Setting



$$q_{xy} := P'_{xy}(0)$$

$$= \begin{cases} -q_x & y = x \\ q_x Q_{xy} & y \neq x \end{cases}$$

"infinitesimal params"

$$q_{xx} := -q_x .$$

Rate Matrix

$$\begin{bmatrix} q_{11} & q_{12} & q_{13} & \cdots & q_{1d} \\ q_{21} & q_{22} & q_{23} & \cdots & q_{2d} \\ \vdots & & \ddots & & \\ & & & & \end{bmatrix}$$

Row Sum $= 0$

$$q_{xx} = -q_x$$

$$q_{xy} = q_x Q_{xy}$$

q_{xy} has the interpretation of the rate at which the system is leaving x to y.

We get:

$$P'_{xy}(t) = \sum_z q_{xz} P_{zy}(t), \quad t \geq 0.$$

Kolmogorov Backward Eqn.

Another Way:

$$P_{xy}(t+s) = \sum_{z} P_{xz}(t)\, P_{zy}(s)$$

Differentiate wrt s and set to 0:

KFE

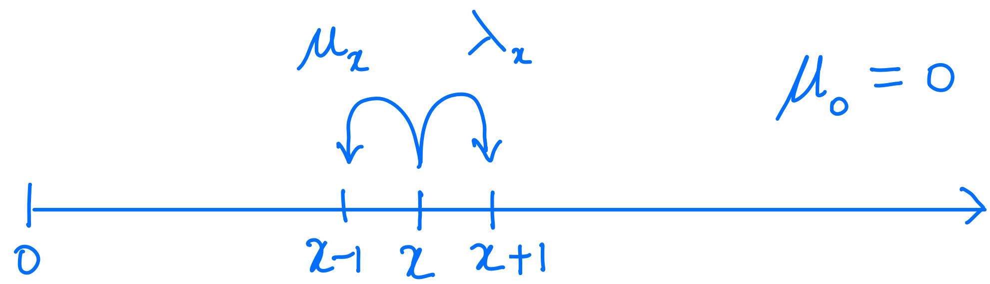
$$P'_{xy}(t) = \sum_{z} P_{xz}(t)\, q_{zy}$$

Differentiate wrt t and set to 0:

KBE

$$P'_{xy}(s) = \sum_{z} q_{xz}\, P_{zy}(s)$$

EXAMPLE (BIRTH DEATH)

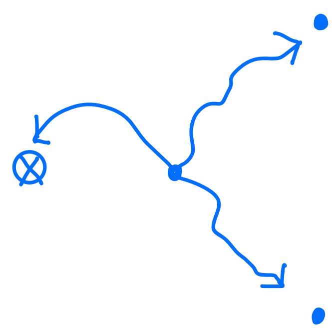


$\mu_0 = 0$

$$q_{xy} = \begin{cases} \lambda_x & y = x+1 \\ \mu_x & y = x-1 \\ -(\lambda_x + \mu_x) & y = x \end{cases}$$

$$Q_{xy} = \begin{cases} \lambda_x / \lambda_x + \mu_x & y = x+1 \\ \mu_x / \lambda_x + \mu_x & y = x-1 \end{cases}$$

EXAMPLE (BRANCHING)



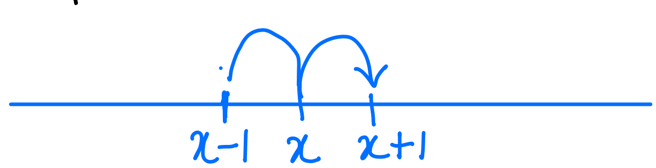
After e_q time, "splits in two," or "dies."

$X(t)$: number of particles at t

x particles present;

$e_{q_1}, e_{q_2}, \ldots, e_{q_x} \sim \exp(q)$

split prob. $= p$



$$q_x = x\,q \qquad \text{(Why?)}$$

$$q_{xy} = \begin{cases} p\,q_x & y = x+1 \\ (1-p)\,q_x & y = x-1 \\ -q_x & y = x \end{cases}$$

Recap

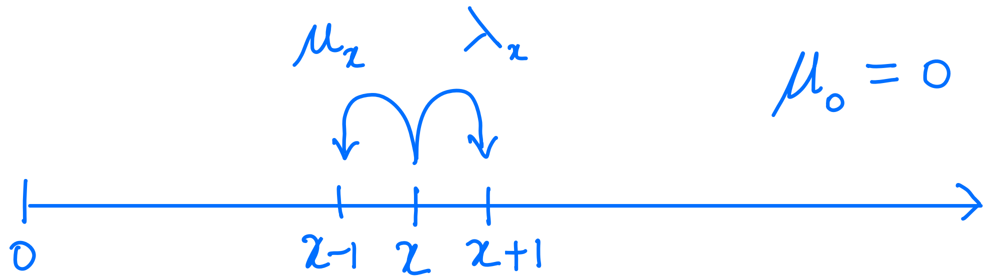
$$P_{xy}(t) = \delta_{xy} e^{-q_x t} + \int_0^t q_x e^{-q_x s} \sum_{z \neq x} Q_{xz} P_{zy}(t-s)\, ds$$

$$P'_{xy}(t) = \sum_z q_{xz} P_{zy}(t)$$

$$= \sum_z P_{xz}(t)\, q_{zy}$$

$$q_{xy} := P'_{xy}(0) = \begin{cases} q_x Q_{xy} & y \neq x \\ -q_x & y = x \end{cases}$$

EXAMPLE (Two State Birth Death)



$\mu_0 = 0$

$$q_{01} = \lambda, \quad q_{00} = -\lambda$$

$$q_{10} = \mu, \quad q_{11} = -\mu$$

$$P_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t};$$

$$P_{10}(t) = \frac{\mu}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}, \quad t \geq 0$$

(How?)

$\boxed{/\!/}$

Let $X(t),\ 0 \le t < \infty$ be a Markov pure jump process with state space S.

$$T_y := \min\{t \geq \tau_1 : X(t) = y\} \qquad \text{—— (1)}$$

Where τ_1 is the time of first jump.

Understand why (1) includes τ_1 in the definition.

$$f_{xy} := P\left(T_y < \infty \mid X(0) = x \right)$$

A state $y \in S$

is recurrent if $f_{yy} = 1$;

is transient if $f_{yy} < 1$.

The process is irreducible

if $f_{xy} > 0 \quad \forall\, x, y \in S$.