$\boxed{\text{Second Order Processes}}$

$$\{ X(t),\ t \in T \}$$

T: "index set"

$\{X(t), t \in T\}$ is second order

if $\mathbb{E}[X^2(t)] < \infty \quad \forall\, t \in T.$

Mean Function

$$\mu_X(s) := \mathbb{E}\left[X(s) \right], \quad s \in T$$

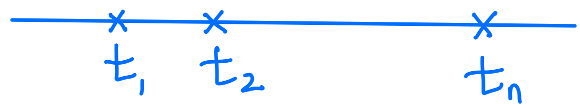
Covariance Function

$$\begin{aligned}
\gamma_X(s,t) &:= \mathrm{Cov}\left(X(s), X(t) \right) \\
&= \mathbb{E}\left[\left(X(s) - \mu_X(s) \right) \left(X(t) - \mu_X(t) \right) \right] \\
&= \mathbb{E}\left[X(s) X(t) \right] - \mu_X(s) \mu_X(t).
\end{aligned}$$

SYMMETRIC

$$\gamma_X(s,t) = \gamma_X(t,s)$$

PSD



$$0 \leq \mathrm{Var} \sum_{j=1}^{n} b_j X(t_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} b_i b_j \gamma_X(t_i, t_j)$$

$$\forall\, t_1, t_2, \dots, t_n$$
$$b_1, b_2, \dots, b_n$$

Conclude, $\gamma_X(\cdot,\cdot)$ is positive semi definite.

Assume $T = (-\infty, \infty)$.

A second order process $\{X(t), t \in T\}$ is covariance stationary if

$$Y(t) = X(t + \tau)$$

and $X(t)$ have the same covariance function $\forall \tau$.

Cov. Stationarity versus Strict Stationarity

A stochastic process $\{X(t), t \in T\}$ is strictly stationary if

$$Y(t) = X(t+\tau)$$

and $X(t)$ have the same joint distbn. functions.

For a covariance stationary process,

(i) $\mu_X(t) = \text{constant}$

(ii) $$\begin{aligned}\gamma_X(s,t) &= \gamma_X(0, t-s) \\ &= \gamma_X(s-t, 0) \\ &= \gamma_X(0, s-t) \\ &= \gamma_X(0, |s-t|)\end{aligned}$$

$$\boxed{\gamma_X(s,t) \text{ depends only on } |s-t|.}$$

(ii) $\Rightarrow \operatorname{Var}(X(t)) = \gamma(t,t) = \text{constant}.$

NOTE:

- "Cov. Stationary," "Second-Order Stationary," "Weakly Stationary" are all synonyms — they mean the same.

- Cov. Stat. $\not\Longleftrightarrow$ Strict Stat. in general.

  Strict. Stat $\Longrightarrow$ Cov. Stat

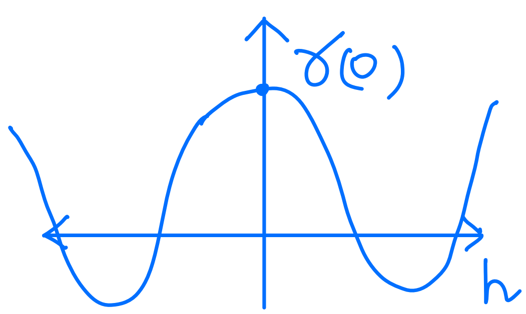
  if $X(t),\ t \in T$ is second order.

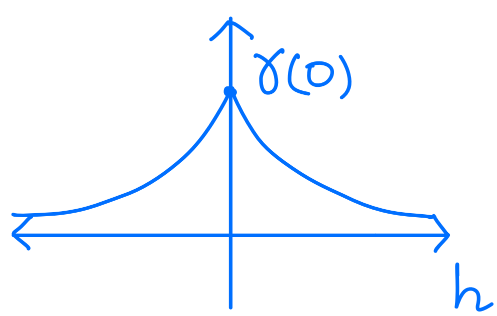
So, we ease notation and write

"lag"

$$\gamma_X(h), \quad h \in \mathbb{R}$$

recognizing that γ_X is symmetric about the origin.





$\gamma_X(h) \not\to 0$ as $h \to \infty$, in general.

$$\rho_X(h) := \gamma_X(h) / \gamma_X(0)$$

"lag-h correlation"

EXAMPLE

The Poisson process $\{X(t), t \in \mathbb{R}\}$ is <u>not covariance stationary</u>.

However,

$$Y_t = X(t+1) - X(t), \quad t \in \mathbb{R}$$

is covariance stationary with

$$\gamma_Y(h) = \begin{cases} \lambda(1-|h|), & h \leq 1 \\ 0 & h > 1 \end{cases}$$

CROSS COVARIANCE

For two second order processes $X(t), t \in \mathbb{R}$ and $Y(t), t \in \mathbb{R}$

$$\gamma_{XY}(s, t) := \operatorname{Cov}\big(X(s), Y(t)\big)$$

Note that symmetry is not presumed:

$$\gamma_{XY}(s,t) \neq \gamma_{XY}(s,t)$$

If $\gamma_{XY}(s,t) = 0$, then

$$\gamma_{X+Y}(s,t) = \gamma_X(s,t) + \gamma_Y(s,t).$$

(How?)

GAUSSIAN PROCESSES

A stochastic process $\{X(t), t \in T\}$ is called a Gaussian process if

$$\left(X(t_1), X(t_2), \ldots, X(t_n)\right)$$

is (multivariate) normal for every $t_1, t_2, \ldots, t_n \in \mathbb{R}$, $n < \infty$.

Alternately and equivalently:

$X(t), t \in T$ is Gaussian if and only if

$$\lambda_1 X(t_1) + \lambda_2 X(t_2) + \cdots + \lambda_n X(t_n)$$

is normal (Gaussian) $\forall\, t_1, t_2, \ldots t_n \in T$, $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$, $n < \infty$.

- Gaussian processes are not necessarily cov. stationary.

(Why?)

- If a Gaussian process is cov. stationary, then it is also strictly stationary.

(Why?)

A Gaussian process is fully specified by its mean and covariance functions.

(All fin. dim. distbns. are decided)

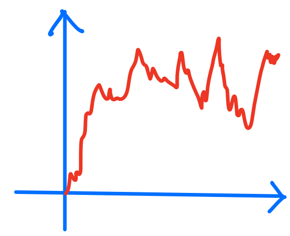
EXAMPLE

$$X(t) = Z_1 \cos 2\pi f_1 t + Z_2 \sin 2\pi f_2 t$$

where Z_1, Z_2 are independent and normal. Then, $X(t)$ is a Gaussian process.

Brownian Motion (Wiener Process)

A stochastic process $X(t)$, $t \in \mathbb{R}$ is called a Wiener process if



(i) $W(0) = 0$

(ii) independent increments, that is, for $t_1 \leq t_2 \leq \cdots \leq t_n$

$(W(t_1) - W(0)), (W(t_2) - W(t_1)) \ldots,$

$(W(t_n) - W(t_{n-1}))$ are independent.

(iii) $W(t) - W(s) \sim N(0, \sigma^2(t-s)),$

$t \geq s.$

For our purposes, a constant random variable is normal with zero variance

A Wiener process $W(t)$, $t \in \mathbb{R}$ is a Gaussian process with

$$\mu_W(t) = 0.$$

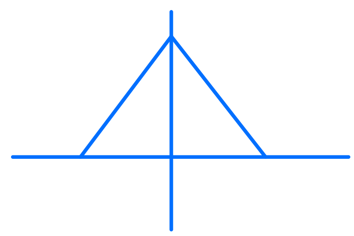
$$\gamma_W(s,t) = \begin{cases} \sigma^2 \min(s,t), & st \geq 0 \\ 0, & st < 0 \end{cases}$$

(How?)

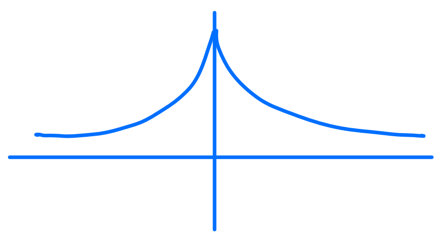
Calculate the covariance function.

(i) $X(t) = \dfrac{W(t+\varepsilon) - W(t)}{\varepsilon}$, $\varepsilon > 0$, $t \in \mathbb{R}$

"first diff. process"



(ii) $X(t) = e^{-\alpha t} W(e^{2\alpha t})$; $t \in \mathbb{R}$, $\alpha > 0$



(iii) $B(t) = W(t) - t\,W(1)$, $t \in [0,1]$

"Brownian bridge"

Assume henceforth that T is an interval of positive length.

Minimal Assumptions:

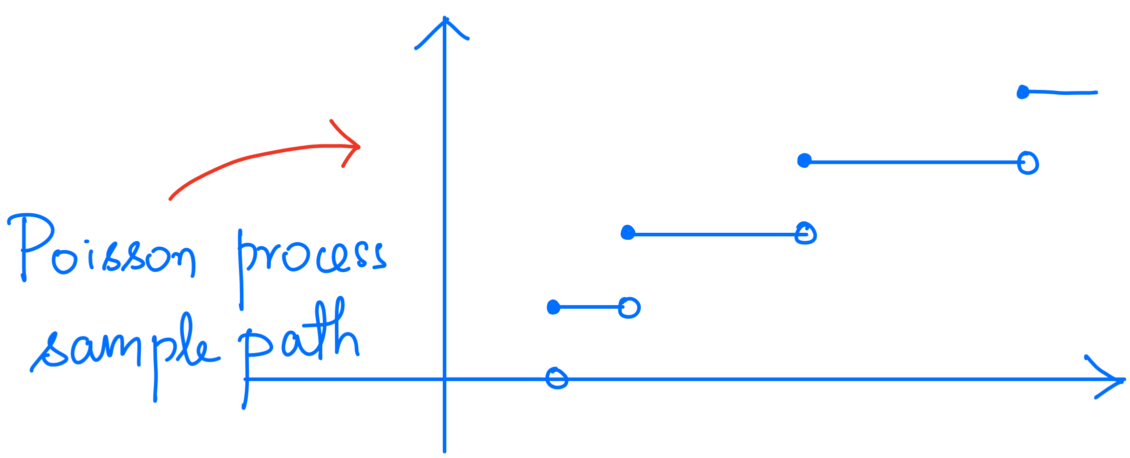
(I) $\mu_X(t)$ is continuous in $t \in T$

(II) $\gamma_X(s,t)$ is continuous in s and $t \in T$.

Can we connect (I) & (II) to continuity of $\{X(t), t \in T\}$?

Notice that (I) and (II) do not guarantee the continuity of the sample path $X(\cdot, \omega)$, e.g. Poisson process.

understand this object.



$\boxed{\text{Theorem}}$

(a) Suppose $\{X(t), t \in T\}$ is second order and satisfies (I) & (II). Then, $X(t), t \in T$ is continuous in mean square:

$$\lim_{s \to t} \mathbb{E}\left[\left(X(s) - X(t)\right)^2\right] = 0.$$

(Proof?)

(contd...)

(b) If $\{X(t), t \in T\}$ and $\{Y(t), t \in T\}$ satisfy (I) and (II), then

$\gamma_{XY}(s,t)$ is continuous in $s, t \in T$.

(Proof?)

What questions are we looking to answer?

- when are $I_1 := \int_a^b f(t)\, X(t)$,

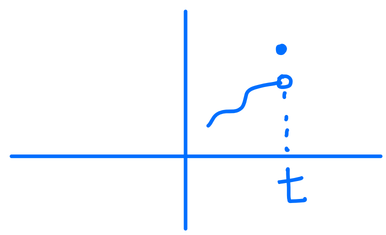
$$I_2 := \int_a^b f(t)\, X(t) \times \int_c^d g(t)\, X(t)$$

well defined?

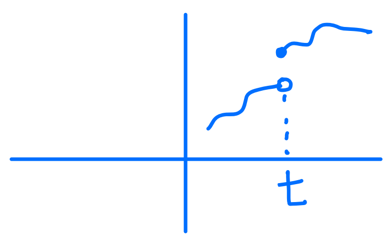
- what is $\mathbb{E}[I_1]$,

$\mathrm{Cov}(I_1, I_2)$?

III $\lim_{s \to t^-} X(s,\omega)$ exists



IV $\lim_{s \to t^+} X(s,\omega) = X(t,\omega)$



V $X(t,\omega)$ has only a finite number of discontinuities in any closed bounded interval.

- the conditions in $\underline{\overline{\text{III}}}$ and $\underline{\overline{\text{IV}}}$ characterize <u>cadlàg</u> functions

- in many settings $X(t,\omega)$ is <u>continuous</u> in t except for ω in a set of prob. zero.

$\boxed{\text{Theorem}}$

Suppose I - $\underline{\mathrm{V}}$ are satisfied, and f, g are piecewise continuous real valued functions in T.

$$(i) \quad \mathbb{E}\left[\int_a^b f(t)\, X(t, \omega)\right] = \int_a^b f(t)\, \mu_X(t)$$

(ii) $$\mathbb{E}\left[\int_a^b f(t)\, X(t,\omega) \times \int_c^d g(t)\, X(t,\omega) \right]$$

$$= \int_a^b f(t) \int_c^d g(s)\, \mathbb{E}\left[X(s,\omega)\, X(t,\omega) \right] ds\, dt$$

Furthermore, if $X(t)$, $t \in T$ is a Gaussian process, then $\int_a^b f(t)\, X(t)$ is (univariate) normal.

EXAMPLE

Let $W(t)$, $-\infty < t < \infty$ be the Wiener process (with cont. sample paths). Then,

(i) $\int_0^1 W(t)\,dt \sim N(0, \sigma^2/3)$.

(ii) $\mathrm{Corr}\left(W(t), \int_0^1 W(s)\,ds\right) = \,?$

(How?)

$\square$

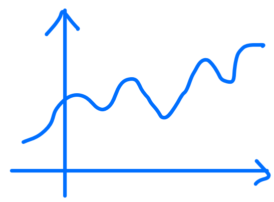
More generally, if f and g are continuously differentiable in $[a, b]$, $a, b \in \mathbb{R}$, then

$$\mathbb{E}\left[\int_a^b f'(t)\big(W(t)-W(a)\big) \times \int_a^b g'(t)\big(W(t)-W(a)\big)\right]$$

$$= \sigma^2 \int_a^b \big(f(t)-f(b)\big)\big(g(t)-g(b)\big).$$

(How?)

Differentiation



Suppose $X(t), t \in T$ is second order satisfying (I)–(II), and $Y(t), t \in T$ is another sec. ord. process satisfying (I)–(IV) such that $\forall\, t_0 \in T$

$$X(t) - X(t_0) = \int_{t_0}^{t} Y(s)\, ds, \quad t \in T.$$

$Y(t), t \in T$ is called the derivative of $X(t), t \in T$ and denoted $X'(t)$.

We would like to know more about $X'(t),\ t \in T$.

For example:

Q.1. How do $\mu_{X'}(t)$, $\gamma_{X'}(s,t)$ relate to $\mu_X(t)$, $\gamma_X(s,t)$?

Q.2 If $X(t),\ t \in T$ is cov. stat what can we say about $X'(t),\ t \in T$?

Q.3 Can we "estimate" $X'(t)$, say by finite differencing $X(t)$?

Q.4 What if $X(t),\ t \in T$ is Gaussian?

Let's answer Q.1

$$\mu_{X'}(t) = \mu'_X(t) \quad , \quad t \in T$$

$$\gamma_{X'}(s,t) = \frac{\partial^2}{\partial s \partial t} \gamma_X(s,t),$$

$$s,t \in T.$$

Proof:

Let's answer Q. 2

If $X(t), t \in T$ is cov. stat.

then,

$$\gamma_{X'}(s,t) = \frac{\partial^2}{\partial s \partial t} \gamma_X(s,t)$$

$$= -\gamma_X''(s-t)$$

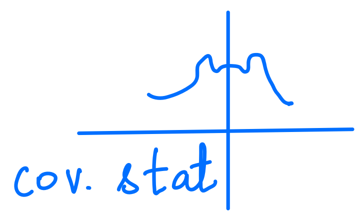
(How?)

$\Rightarrow X'(t), t \in T$ is cov. stat.

In fact, the above shows X and X' are

uncorrelated:

$$\gamma_{XX'}(s,t) = \frac{\partial}{\partial t}\gamma_X(s,t)$$



$$\Rightarrow \gamma_{XX'}(t,t) = -\gamma_X'(0) = 0$$

(How?)

Let's answer Q.3

Let $X(t),\ t \in T$ be a differentiable second order process.

Consider the natural estimator

$$\hat{X}(t,\varepsilon) := \frac{X(t+\varepsilon) - X(t)}{\varepsilon}$$

of $X'(t)$.

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\left(\frac{X(t+\varepsilon)-X(t)}{\varepsilon} - X'(t)\right)^2\right] = 0$$

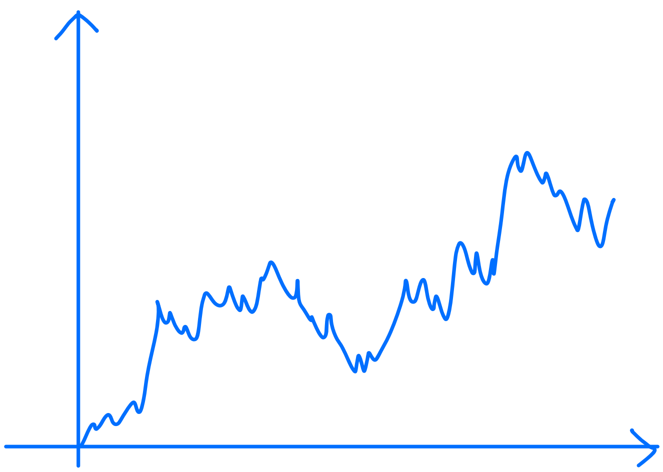
(Proof?)

Let's answer Q.4

Suppose $X(t), t \in T$ is a differentiable second order process that is also Gaussian, then $X'(t), t \in T$ is also Gaussian.

A proof is outside the scope.

The Wiener process $W(t),\ t \in T$ is continuous everywhere but differentiable nowhere.



(It is easy to see that $W(t),\ t \in T$ is not differentiable)

Even though $W'(t)$ does not exist, notice:

$$\frac{d}{dt}\int_t^{t+\varepsilon} W(s)\,ds = W(t+\varepsilon) - W(t)$$

Why?

And, hence

$$\int_a^b f(t)\,\frac{1}{\varepsilon}\Big(W(t+\varepsilon) - W(t)\Big)$$

$$= \int_a^b \underbrace{f(t)}_{u}\,\frac{d}{dt}\underbrace{\frac{1}{\varepsilon}\int_t^{t+\varepsilon} W(s)\,ds}_{v}$$

$$= \underbrace{f(t)}_{u}\underbrace{\frac{1}{\varepsilon}\int_t^{t+\varepsilon} W(s)\,ds}_{v}\Bigg|_a^b$$

$$- \int_a^b f'(t)\underbrace{\frac{1}{\varepsilon}\int_t^{t+\varepsilon} W(s)\,ds}_{v}\,dt$$

Now send $\varepsilon \to 0$ to get

$$\lim_{\varepsilon\to 0}\int_a^b f(t)\,\frac{1}{\varepsilon}\Big(W(t+\varepsilon) - W(t)\Big)$$

$$= f(b)W(b) - f(a)W(a)$$

$$- \int_a^b f'(t)\,W(t)\,dt$$

$$\lim_{\varepsilon \to 0} \int_a^b f(t) \frac{1}{\varepsilon} \Big(W(t+\varepsilon) - W(t) \Big)$$

$$= f(b) W(b) - f(a) W(a) - \int_a^b f'(t)\, W(t)\, dt$$

Notice that this looks like integration by parts!

So, we define

$$\int_a^b f(t)\, W'(t)\, dt$$

$$:= \lim_{\varepsilon \to 0} \int_a^b f(t) \frac{1}{\varepsilon} \Big(W(t+\varepsilon) - W(t) \Big)$$

$$= f(b) W(b) - f(a) W(a) - \int_a^b f'(t)\, W(t)\, dt$$

$$\int_a^b f(t)\, W'(t)\, dt$$

is sometimes written as

$$\int_a^b f(t)\, dW(t)$$

It seems clear (although no proof will be given) that

$$\int_a^b f(t)\, dW(t)$$

is (univariate) Gaussian.

Why is any of this important?

Because, a "stochastic differential equation" (SDE)

$$a_0 X''(t) + b_0 X'(t) + c_0 X(t) = W'(t)$$

is well defined!