ASSIGNMENT IV, STAT 532 (Pasupathy), Fall 2020

1. Determine the transition matrix for the following Maarkov chains.

   (a) Consider a sequence of n tosses of a coin with the probability of "heads" p. At time n (after n tosses) the state of the process is the number heads minus the number of tails.

   (b) N black balls and N white balls are placed in two urns so that each urn contains N balls. At each step, one ball is selected at random from each urn and the two balls are exchanged. The state of the system is the number of white balls in the first urn.

2. Let's recall the one-dimensional random walk discussed in class. $\{X_k, k \geq 0\}$ is a Markov chain with $X_k \in \mathbb{Z}$ (set of integers) and having transition function

$$\pi(x, y) = \begin{cases} p_r & y = x + 1; \\ p_\ell = 1 - p_r & y = x - 1; \\ 0 & y \notin \{x + 1, x - 1\}. \end{cases}$$

   (Loosely, in the one-dimensional random walk, the chain "steps to the right" with probability p_r, and "to the left" with probability p_ℓ.) Show that if $p_r = p_\ell = \frac{1}{2}$, then all states $x \in \mathbb{Z}$ are recurrent; otherwise, all states $x \in \mathbb{Z}$ are transient.

3. Now consider the two-dimensional random walk where $\{X_k, k \geq 0\}$ is a Markov chain $X_k \in \mathbb{Z} \times \mathbb{Z}$ (two-dimensional integer lattice), having transition function

$$\pi((x_1, y_1), (x_2, y_2)) = \begin{cases} p_r & (x_2, y_2) = (x_1 + 1, y_1); \\ p_\ell & (x_2, y_2) = (x_1 - 1, y_1); \\ p_u & (x_2, y_2) = (x_1, y_1 + 1); \\ p_d & (x_2, y_2) = (x_1, y_1 - 1); \\ 0 & \text{otherwise}, \end{cases}$$

   where $p_r + p_\ell + p_u + p_d = 1$ and $p_r, p_\ell, p_u, p_d \geq 0$. (Loosely, in the two-dimensional random walk, the chain steps to the right, left, up, and down with the respective probabilities p_r, p_ℓ, p_u and p_d.) Show that the chain is recurrent if $p_r = p_\ell = p_u = p_d = \frac{1}{4}$.

Hint: In the second and third problems, it is difficult to show that a state z is recurrent or transient directly, from the definition. Instead, since every state communicates with every other, use the fact that a state z is recurrent if and only if

$$\mathbb{E}[N(z) \,|\, X_0 = x_0] = \sum_{n=1}^{\infty} P(X_n = z \,|\, X_0 = x_0) = \infty.$$

Also, Stirling's beautiful formula $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$, where $a_n \sim b_n$ means $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.