ASSIGNMENT VI, STAT 532 (Pasupathy), Fall 2020

1. Let π_0 and π_1 be distinct stationary distributions for a Markov chain.

   (a) Show that for $0 \le \alpha \le 1$, the function π_α defined by

   $$\pi_\alpha(x) = (1-\alpha)\pi_0(x) + \alpha\pi_1(x), \quad x \in \mathcal{S}$$

   is a stationary distribution.

   (b) Show that distinct values of α determine distinct stationary distributions π_α.

2. Let $\{X_n, n \ge 0\}$ be a positive recurrent irreducible birth and death chain, and suppose X_0 has the stationary distribution π. Find $P(X_0 = y \mid X_1 = x)$ for $x, y \in \mathcal{S}$.

3. Suppose that ξ_n particles are added to a box at times $n = 1, 2, \ldots$, where $\xi_n, n \ge 1$, are independent and have a Poisson distribution with common parameter λ. Suppose that each particle in the box at time n, independently of all the other particles in the box and independently of how particles are added to the box, has probability $p < 1$ of remaining in the box at time $n+1$ and probability $q = 1-p$ of being removed from the box at time $n+1$. Let X_n denote the number of particles in the box at time n.

   (a) If X_0 has a Poisson distribution with parameter t, then find the distribution of X_n.

   (b) Find $\mathbb{E}[X_n \mid X_0 = x]$.

4. Consider a Markov chain on the nonnegative integers having transition function P given by $P(x, x+1) = p$ and $P(x, 0) = 1 - p$ where $0 < p < 1$. Show that this chain has a unique stationary distribution π. Find π.

5. Consider the Ehrenfest chain. Suppose that initially all of the balls are in the second box. Find the expected amount of time until the system returns to that state.