More Practice Problems, STAT 532 (Pasupathy)

1. Let a Markov chain have r states. Prove the following.
   (a) If a state can be reached from j, then it can be reached in $r-1$ states or less.
   (b) If j is a recurrent state, there exists $\alpha(0 < \alpha < 1)$ such that for $n > r$, the probability that first return to state j occurs after n transitions is $\leq \alpha^n$.

2. Suppose two distinguishable fair coins are tossed simultaneously and repeatedly. An account of the tallies of heads and tails are recorded. Consider the event E_n that at the n-th toss the cumulative number of heads on both tallies are equal. Relate this event to the recurrence time of a given state for a symmetric random walk on the integers.

3. A telephone exchange has m channels. Calls arrive according to a Poisson process with parameter λ, and a call is accepted if there is an empty channel. If there is no empty channel, then an arriving call is lost. The duration of each call is exponential with parameter μ. The lifetimes of separate calls are independent. Find the stationary probabilty of the number of busy channels.

4. A birth and death process is called a *linear growth process* if $\lambda_n = \lambda n + a$ and $\mu_n = \mu n$ with $\lambda > 0, \mu > 0, a > 0$. (Such a model is is very common and useful in the context of biological processes.) Suppose that for a linear growth process, $\lambda = \mu$. Prove that

$$u(t) = P(X(t) = 0 \,|\, X(0) = 1)$$

   satisfies

$$u(t) = \frac{1}{2}\int_0^t 2\lambda e^{-2\lambda\tau}\,d\tau + \frac{1}{2}\int_0^t 2\lambda e^{-2\lambda\tau}\left(u(t-\tau)^2\right)^2\,d\tau.$$

   Show that $u(t)$ satisfies the Ricatti equation

$$u'(t) + 2\lambda u(t) = \lambda + \lambda u^2(t).$$

5. Set

$$X(t) = \int_t^{t+1} (W(s) - W(t))\,ds, \quad -\infty < t < \infty.$$

   Show that this is a covariance stationary process with mean zero and find $\gamma_X(t), -\infty < t < \infty$.

6. Let $X(t), -\infty < t < \infty$ be a covariance stationary process satisfying assumption (ii)-(v) laid out in class. Obtain an expression for

$$\text{Var}\left(\frac{1}{T}\int_0^T X(t)\,dt\right)$$

   in terms of $\gamma_X(\cdot)$.

7. Let $X(t), -\infty < t < \infty$ be a covariance stationary process satisfying assumption (ii)-(v) laid out in class, having a constant but unknown mean μ, and and covariance function

$$\gamma_X(t) = \alpha e^{-\beta|t|}, \quad -\infty < t < \infty,$$

where α and β are constants. Set

$$\overline{X} = \frac{1}{T} \int_0^T X(t)\, dt.$$

Show that $\overline{X}$ is an unbiased estimator of μ and calculate $\mathrm{Var}(\overline{X})$. Set

$$\hat{\mu} = \frac{X(0) + X(T) + \beta \int_0^T X(t)\, dt}{2 + \beta T}.$$

Show that $\hat{\mu}$ is an unbiased estimator of μ and calculate $\mathrm{Var}(\hat{\mu})$.