ASSIGNMENT II, MA/STAT 532 (Pasupathy), Fall 2022

Assigned on Saturday, September 3
Due Saturday, September 10

(Full credit if you complete any five problems.)

1. (D) Ron, Sue, and Ted arrive at the beginning of a professor's office hours. The amount of time they will stay is exponentially distributed with means of 1, 1/2, and 1/3 h.
   (a) What is the expected time until only one student remains?
   (b) For each student, find the probability that they are the last student left.
   (c) What is the expected time until all three students depart?
2. (D) The number of hours between successive trains is T which is uniformly distributed between 1 and 2. Passengers arrive at the station according to a Poisson process with rate 24 per hour. Let X denote the number of people who get on a train. Find $\mathbb{E}[X]$ and $\text{Var}(X)$.
3. (D) Consider a Poisson process with rate λ and let L be the time of the last arrival in the interval $[0, t]$, with $L = 0$ if there was no arrival.
   (a) Compute $\mathbb{E}[t - L]$.
   (b) Compute $\lim_{t\to\infty} \mathbb{E}[t - L]$.
4. (D) Let $\{S_t, t \geq 0\}$ be the price of a stock at time t and suppose that at arrival times of a Poisson process $\{X_t, t \geq 0\}$ with rate λ, the price is multiplied by a random variable $Y_j > 0$ with mean μ and variance σ^2, that is,
$$S_t = S_0 \prod_{j=1}^{X_t} Y_j,$$
   where the product is 1 if $X_t = 0$. Find $\mathbb{E}[S_t]$ and $\text{Var}(S_t)$. (Assume $Y_j, j = 1, 2 \ldots$ are iid and independent of $\{X_t, t \geq 0\}$.)
5. (R) Suppose passengers arrive at a train station according to a Poisson process $\{X_t, t \geq 0\}$ having parameter λ. Two trains are scheduled to depart the station. The first of these will depart at some chosen time $t \in [0, T]$, and carry all passengers arriving before time t; the second train will depart at time T, carrying all remaining passengers. What value of t minimizes the total expected waiting time across passengers?
6. Let $\{X_t, t \geq 0\}$ be a homogeneous Poisson process with parameter λ. Find the density function of the random variable $(S_1, S_2, \ldots, S_{X_t})|X_t = n$, where S_j denotes the j-th arrival time.

Supplemental Problems

1. Traffic on State Street a few miles west of campus follows a Poisson process with rate $\lambda > 0$. Deer run of the woods all the time to cross the road. A crossing deer needs a "time gap" of x secs between cars in order to safely cross the road. What is the probability of a collision?

2. In the previous problem, what is the expected amount of time the deer will have to wait before he/she can safely cross the road?

3. Ellen catches fish at times of Poisson process with rate 2 per hour. Forty percent of the fish are salmon, while 60 percent of the fish are trout. What is the probability that she will catch exactly one salmon and two trout if she fishes for 2.5 hours?

4. An insurance company feels that each of its policy holders has a rating value and that a policy holder having rating value λ will make claims at times distributed according to a Poisson process with rate λ, when time is measured in years. The firm also believes that rating values vary from policyholder to policyholder, with the probability distribution of the value of a new policyholder being uniformly distributed over $(0, 1]$. Given that a policyholder has made n claims in the first t years, what is the conditional distribution of the time until the policyholder's next claim?

5. (The Coupon Collecting Problem) There are m different types of coupons. Each time a person collects a coupon, it is, independently of ones previously obtained, a type j coupon with probability p_j, so that $\sum_{j=1}^{m} p_j = 1$. Let N denote the number of coupons one needs to collect in order to have a complete collection of at least one of each type. Find $\mathbb{E}[N]$.

1. $X_1 \to$ Ron

$X_2 \to$ Sue

$X_3 \to$ Ted

$X_{(1)}, X_{(2)}, X_{(3)}$ are order statistics

with $X_{(1)} = \text{Min}(X_1, X_2, X_3)$

and $X_{(3)} = \text{Max}(X_1, X_2, X_3)$.

(a) $\mathbb{E}[X_{(2)}]$

$$= \int_0^\infty P(X_3 \leq x < X_1) f_{X_2}(x)\,dx$$

$$+ \int_0^\infty P(X_1 \leq x < X_3) f_{X_2}(x)\,dx$$

$$+\int_0^\infty P(X_2 \le x < X_3)\, f_{X_1}(x)\, dx$$

$$+\int_0^\infty P(X_3 \le x < X_2)\, f_{X_1}(x)\, dx$$

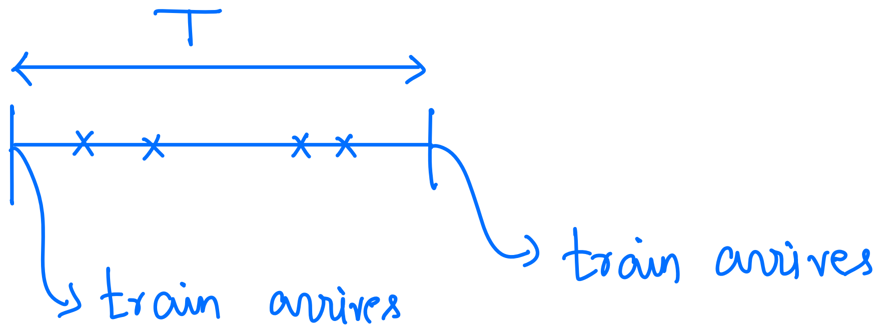
$$+\int_0^\infty P(X_2 \le x < X_1)\, f_{X_3}(x)\, dx$$

$$+\int_0^\infty P(X_1 \le x < X_2)\, f_{X_3}(x)\, dx$$

(b) Find $P(X_j = \max(X_1, X_2, X_3))$ for each $j = 1, 2, 3$.

(c) Find $\mathbb{E}[X_{(3)}]$, similar to (a).

2.



$$\mathbb{E}[X] = \mathbb{E}_T \, \mathbb{E}[X \mid T = t]$$

$$= \int_1^2 24t \, dt = 36$$

$$\mathrm{Var}(X \mid T) = 24T$$

$$\mathbb{E}[X \mid T] = 24T$$

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X \mid T)] + \mathrm{Var}(\mathbb{E}[X \mid T])$$

$$= 24\,\mathbb{E}[T] + 24^2\,\mathrm{Var}(T).$$

③

(a) $S_1, S_2, \ldots, S_n$ arrival times.

$$\mathbb{E}[t - L] = \mathbb{E}[t - S_{X_t}]$$

$$= \mathbb{E}\left[\mathbb{E}\left[t - S_{X_t} \mid X_t = n\right]\right]$$

Recall that

$$(S_1, S_2, \ldots, S_n \mid X_t = n)$$

$$\overset{d}{=} U_{(1)}, U_{(2)}, \ldots U_{(n)}$$

where $U_j \overset{iid}{\sim} \mathrm{Unif}(0, t)$

So,

$$S_{X_t} \mid X_t = n \overset{d}{=} U_{(n)}$$

$$\mathbb{E}\left[t - S_{X_t} \mid X_t = n\right]$$

$$= t - \mathbb{E}\left[S_{X_t} \mid X_t = n\right]$$

$$= t - \int_0^t P(U_{(n)} > u)\, du$$

$$= t - \int_0^t \left(1 - \left(\frac{u}{t}\right)^n\right) du$$

$$= \frac{t}{n+1}$$

Conclude that

$$\mathbb{E}[t - L]$$

$$= \sum_{n=0}^{\infty} \frac{t}{n+1} e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$$= \frac{1}{\lambda} \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{n+1}}{(n+1)!}$$

$$= \frac{1}{\lambda} \left(1 - e^{-\lambda t} \right)$$

Therefore,

$$\lim_{t \to \infty} \mathbb{E}[t - L] = \frac{1}{\lambda} \qquad \blacksquare$$

4.

$$S_t = S_0 \prod_{j=1}^{X_t} Y_j$$

$$\mathbb{E}[S_t] = \mathbb{E}[S_t \mid X_t = n]$$

$$\times e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$$\mathbb{E}[S_t^2] = \mathbb{E}[S_t^2 \mid X_t = n]$$

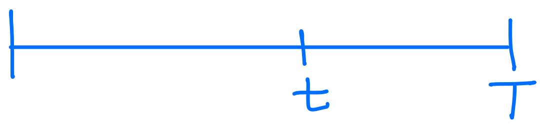
$$\times e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$$\mathbb{E}[S_t \mid X_t = n] = S_0 (\mathbb{E}[Y_j])^n$$

$$\mathbb{E}[S_t^2 \mid X_t = n] = S_0^2 (\mathbb{E}[Y_j])^{2n}$$

Use above expressions to calculat $\mathbb{E}[S_t]$ and $Var(S_t)$.

5.



Expected total wait time : $\mathbb{E}[W]$

$$= \mathbb{E}[W \mid X_T = n]$$

$$\times \; e^{-\lambda T} (\lambda T)^n / n!$$

$$\mathbb{E}\left[W \mid X_T = n\right]$$

$$= n \times \left(\int_0^t \frac{1}{T}(t-u)\,du + \int_t^T \frac{1}{T}(T-u)\,du \right)$$

$$= \frac{n}{T} \times \left(\frac{t^2}{2} + T(T-t) - \frac{1}{2}(T^2 - t^2) \right)$$

$$= \frac{n}{T} \left(t^2 - Tt + \frac{T^2}{2} \right)$$

$$\mathbb{E}[W] = \left(t^2 - Tt + \frac{T^2}{2} \right) n \sum_{n=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!}$$

We see that $\mathbb{E}[W]$ is minimized at $t = T/2$.

6. $(S_1, S_2) \mid X_t = n$ was worked out in detail in class.

Extend to $(S_1, S_2, \ldots S_n) \mid X_t = n$.