ASSIGNMENT IV, STAT 532 (Pasupathy), Fall 2022

1. Determine the transition matrix for the following Markov chains.

   (a) Consider a sequence of n tosses of a coin with the probability of "heads" p. At time n (after n tosses) the state of the process is the number heads minus the number of tails.

   (b) N black balls and N white balls are placed in two urns so that each urn contains N balls. At each step, one ball is selected at random from each urn and the two balls are exchanged. The state of the system is the number of white balls in the first urn.

2. Consider the two-state Markov chain with the one-step transition probability matrix $P(1,1) = 1-p, P(1,2) = p, P(2,1) = q$, and $P(2,2) = 1-q$, where $p+q > 0$. Find P^n.

3. Let $X_n, n \geq 0$ be the two-state Markov chain.

   (a) Find $P(T_0 = n|X_0 = 0)$

   (b) Find $P(T_1 = n|X_0 = 0)$.

4. Let x and y be distinct states of a Markov chain having $d < \infty$ states and suppose that x leads to y. Let n_0 be the smallest positive integer such that $P^{n_0}(x,y) > 0$ and let $x_1, x_2, \ldots, x_{n_0-1}$ be states such that

$$P(x, x_1)P(x_1, x_2) \cdots P(x_{n_0-2}, x_{n_0-1})P(x_{n_0-1}, y) > 0.$$

   (a) Show that $x, x_1, \ldots, x_{n_0-1}, y$ are distinct states.

   (b) Use (a) to show that $n_0 \leq d-1$.

   (c) Conclude that $P(T_y \leq d-1|X_0 = x) > 0$.

5. Let $X_n, n \geq 0$ be a Markov chain whose state space $\mathcal{S}$ is a subset of $\{0, 1, 2, \ldots\}$ and whose transition probability matrix P is such that

$$\sum_y yP(x,y) = Ax + B, \quad x \in \mathcal{S},$$

   for some constants A and B.

   (a) Show that $\mathbb{E}[X_{n+1}] = A\mathbb{E}[X_n] + B$.

   (b) Show that if $A \neq 1$, then

$$\mathbb{E}[X_n] = \frac{B}{1-A} + A^n \left(\mathbb{E}[X_0] - \frac{B}{1-A} \right).$$

6. Let y be a transient sate. Show that for all x,

$$\sum_{n=0}^{\infty} P^n(x,y) \le \sum_{n=0}^{\infty} P^n(y,y).$$

7. Show that $\rho_{xy} > 0$ if and only if $P^n(x,y) > 0$ for some positive integer n.

8. Let's recall the one-dimensional random walk discussed in class. $\{X_k, k \ge 0\}$ is a Markov chain with $X_k \in \mathbb{Z}$ (set of integers) and having transition function

$$\pi(x,y) = \begin{cases} p_r & y = x+1; \\ p_\ell = 1 - p_r & y = x-1; \\ 0 & y \notin \{x+1, x-1\}. \end{cases}$$

(Loosely, in the one-dimensional random walk, the chain "steps to the right" with probability p_r, and "to the left" with probability p_ℓ.) Show that if $p_r = p_\ell = \frac{1}{2}$, then all states $x \in \mathbb{Z}$ are recurrent; otherwise, all states $x \in \mathbb{Z}$ are transient.

9. Now consider the two-dimensional random walk where $\{X_k, k \ge 0\}$ is a Markov chain $X_k \in \mathbb{Z} \times \mathbb{Z}$ (two-dimensional integer lattice), having transition function

$$\pi((x_1,y_1),(x_2,y_2)) = \begin{cases} p_r & (x_2,y_2) = (x_1+1, y_1); \\ p_\ell & (x_2,y_2) = (x_1-1, y_1); \\ p_u & (x_2,y_2) = (x_1, y_1+1); \\ p_d & (x_2,y_2) = (x_1, y_1-1); \\ 0 & \text{otherwise}, \end{cases}$$

where $p_r + p_\ell + p_u + p_d = 1$ and $p_r, p_\ell, p_u, p_d \ge 0$. (Loosely, in the two-dimensional random walk, the chain steps to the right, left, up, and down with the respective probabilities p_r, p_ℓ, p_u and p_d.) Show that the chain is recurrent if $p_r = p_\ell = p_u = p_d = \frac{1}{4}$.

Hint: In the second and third problems, it is difficult to show that a state z is recurrent or transient directly, from the definition. Instead, since every state communicates with every other, use the fact that a state z is recurrent if and only if

$$\mathbb{E}[N(z) \,|\, X_0 = x_0] = \sum_{n=1}^{\infty} P(X_n = z \,|\, X_0 = x_0) = \infty.$$

Also, Stirling's beautiful formula $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, where $a_n \sim b_n$ means $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

1. (a) $X_n = H_n - T_n = 2H_n - n$

where H_n = number of "Heads in n tosses; T_n = number of Tails in n tosses.

$H_n \in \{0, 1, 2, \ldots, n\}$ and hence

$X_n \in \{-n, -n+2, -n+4, \ldots, n\}$

$$P(X_{n+1} = -n+j+1 \mid X_n = -n+j) = p$$

$$P(X_{n+1} = -n+j \mid X_n = -n+j) = 1-p$$

$j = 0, 2, 4, \ldots, n$

(b) At time n, suppose there are W white balls in urn 1.

So there are $N-W$ white balls in urn 2, $N-W$ black balls in urn 1 and W black balls in urn 2.

$$P(X_{n+1} = W \mid X_n = W) = 2\left(\frac{W}{N} \times \frac{N-W}{N}\right)$$

$$P(X_{n+1} = W+1 \mid X_n = W) = \left(\frac{N-W}{N}\right)^2$$

$$P(X_{n+1} = W-1 \mid X_n = W) = \left(\frac{W}{N}\right)^2$$

$$W \in \{N-n, N-n+1, \dots, N+n\}$$

2. Lets find

$$P^n(1,1) = P(X_n = 1 \mid X_0 = 1).$$

$$P^n(1,1) = P(X_n = 1 \mid X_{n-1} = 1)\, P(X_{n-1} = 1 \mid X_0 = 1)$$
$$+ P(X_n = 1 \mid X_{n-1} = 2)\, P(X_{n-1} = 2 \mid X_0 = 1)$$
$$= (1-p)\, P^{n-1}(1,1) + q\left(1 - P^{n-1}(1,1)\right)$$
$$= (1-p-q)\, P^{n-1}(1,1) + q$$

Recursing back, we get

$$P^n(1,1) = (1-p-q)^{n-1} P(1,1) + \sum_{j=0}^{n-2} (1-p-q)^j q$$

$$= (1-p-q)^n \frac{p}{p+q} + \frac{q}{p+q}$$

Similarly,

$$P^n(2,1) = P(X_n = 1 \mid X_{n-1} = 1) P(X_{n-1} = 1 \mid X_0 = 2)$$

$$+ P(X_n = 1 \mid X_{n-1} = 2) P(X_{n-1} = 2 \mid X_0 = 2)$$

$$= (1-p) P^{n-1}(2,1) + q(1 - P^{n-1}(2,1))$$

$$= (1-p-q)^{n-1} P(2,1) + \sum_{j=0}^{n-2} (1-p-q)^j q$$

$$= \frac{q}{p+q} - \frac{q}{p+q}(1-p-q)^n .$$

$\boxtimes$

3. (a) $P(T_0 = n \mid X_0 = 0)$

$$= P(0,1)\,P(1,1)^{n-2}\,P(1,0)$$

(b) $P(T_1 = n \mid X_0 = 0)$

$$= P(0,0)^{n-1}\,P(0,1).$$

$\blacksquare$

4. (a) $x, x_1, x_2, \ldots, x_{n_0-1}, y$ have to be distinct for otherwise we can find a path from x to y that takes fewer than n_0 steps.

(b) Total number of states $= d < \infty$ and so $n_0 \leq d-1$.

(c) $P(T_y \leq d-1 \mid X_0 = x)$

$$\geq P(x, x_1) P(x_1, x_2) \cdots P(x_{n-1}, y) > 0.$$

5. (a) $\mathbb{E}[X_{n+1}] = \mathbb{E}\Big[\mathbb{E}[X_{n+1} \mid X_1]\Big]$

$$= \sum_{x_n} P(X_n = x_n)\,\mathbb{E}[X_{n+1} \mid X_n = x_n]$$

$$= \sum_{x_n} P(X_n = x_n) \sum_{x_{n+1}} x_{n+1} P\left(X_{n+1} = x_{n+1} \middle| X_n = x_n\right)$$

$$= \sum_{x_n} P(X_n = x_n)\left(A x_n + B\right)$$

$$= A\,\mathbb{E}[X_n] + B.$$

(b) Recurse backward and sum.

$\square$

6. For a transient state y,

$$\sum_{n=0}^{\infty} P^n(x,y) = P^0(x,y) + \sum_{n=1}^{\infty} P^n(x,y)$$

$$= \delta_{xy} + \sum_{n=1}^{\infty} P^n(x,y)$$

We know that $\forall\ x \in S$,

$$\sum_{n=1}^{\infty} P^n(x,y) = \frac{\rho_{xy}}{1-\rho_{yy}}$$

Therefore,

$$\sum_{n=0}^{\infty} P^n(x,y) = \delta_{xy} + \frac{\rho_{xy}}{1-\rho_{yy}}$$

$$\leq \frac{1}{1-\rho_{yy}} = \sum_{n=0}^{\infty} P^n(y,y)$$

$\blacksquare$

7. ($\Rightarrow$) If $P^{n_1}(x,y) > 0$, then

$$\rho_{xy} = P(T_y < \infty \mid X_0 = x)$$

$$= P\left(\bigcup_{n=1}^{\infty} I_y(X_n \mid X_0 = x) \right)$$

$$\geq P\left(I_y(X_{n_1} \mid X_0 = x) \right)$$

$$= P^{n_1}(x,y) > 0.$$

($\Leftarrow$) Now suppose $\rho_{xy} > 0$.

Then, we know that

$$G(x,y) = \frac{\rho_{xy}}{1-\rho_{yy}} = \sum_{n=1}^{\infty} P^n(x,y) > 0.$$

Hence, $\exists\, n_1$ such that $P^{n_1}(x,y) > 0$.

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8. Lets calculate $G(0,0)$.

We see that for $n \geq 1$,

$$P(X_{2n+1} = 0 \mid X_0 = 0) = 0$$

and

$$P(X_{2n} = 0 \mid X_0 = 0) = P(S = n)$$

Where $S \sim Bin(2n, 1/2)$.

$$P(S = n) = \binom{2n}{n} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^n$$

$$= \frac{2n!}{n!\,n!} \frac{1}{2^{2n}}$$

From Stirling,

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\,(n/e)^n} = 1$$

So,

$$\frac{2n!}{n!\,n!}\frac{1}{2^{2n}} \sim \frac{\sqrt{\cancel{2\pi}\,\cancel{2n}}\left(\cancel{\frac{2n}{e}}\right)^{\cancel{2n}}}{\sqrt{\cancel{2\pi n}}\left(\cancel{\frac{n}{e}}\right)^{\cancel{n}}\sqrt{\cancel{2}\pi n}\left(\cancel{\frac{n}{e}}\right)^{\cancel{n}}\cancel{2^{2n}}}$$

$$= \frac{1}{\sqrt{\pi n}}$$

Precisely,

$$\lim_{n\to\infty} \frac{\frac{2n!}{n!\,n!}\frac{1}{2^{2n}}}{1/\sqrt{\pi n}} = 1.$$

Since,

$$G(0,0) = \sum_{n=0}^{\infty} P^{2n}(0,0)$$

and

$$\lim_{n\to\infty} \frac{P^{2n}(0,0)}{1/\sqrt{\pi n}} = 1,$$

conclude

$$G(0,0) = \infty,$$

and 0 is a recurrent state.

A similar analysis for any $G(x,x)$.

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9. Similar to the previous problem, let's calculate $G((0,0),(0,0))$.

To return to $(0,0)$ after $2n$ steps, the chain has to make <u>j steps to the right</u>, <u>j steps to the left</u>, <u>$n-j$ steps up</u> <u>and $n-j$ steps down for $j=1,2,\dots,n$.</u>

$$P^{2n}\big((0,0),(0,0)\big) =$$

$$\sum_{j=0}^{n} \binom{2n}{n-j}\binom{n+j}{n-j}\binom{2j}{j} p_u^{\,n-j}\, p_d^{\,n-j}\, p_\ell^{\,j}\, p_r^{\,j}$$

Now use Stirling again to demonstrate that

$$\sum_{n=1}^{\infty} P^{2n}\big((0,0),(0,0)\big) = \infty$$

if $p_\ell = p_r = p_u = p_d = 1/4$.