ASSIGNMENT V, STAT 532 (Pasupathy), Fall 2022

1. A gambler playing roulette makes a series of one dollar bets. He has respective probabilities 9/19 and 10/19 of winning and losing each bet. The gambler decides to quit playing as soon as he either is one dollar ahead or has lost his initial capital of \$1000.

   (a) Find the probability that when he quits playing he will have lost \$1000.

   (b) Find his expected loss.

2. Consider a birth and death chain on the nonnegative integers such that $p_x > 0$ and $q_x > 0$ for $x \geq 1$.

   (a) Show that if $\sum_{y=0}^{\infty} \gamma_y = \infty$, then $\rho_{x0} = 1, x \geq 1$.

   (b) Show that if $\sum_{y=0}^{\infty} \gamma_y < \infty$, then

$$\rho_{x0} = \frac{\sum_{y=x}^{\infty} \gamma_y}{\sum_{y=0}^{\infty} \gamma_y}, \quad x \geq 1.$$

3. A simple model of gas exchange is what is called the *Ehrenfest* chain, described as follows. Suppose we have two boxes labeled I and II, with d balls labeled $1, 2, \ldots, d$. Initially some of the balls are in Box I and others in Box II. An integer is selected at random from $1, 2, \ldots, d$ and the ball with that integer as its label is removed from its box and placed into the other. This procedure is repeated indefinitely with the selections being independent from trial to trial. Let $X_n, n \geq 0$ denote the number of balls in Box I after the n-th trial. Find $\mathbb{E}[X_n | X_0 = x]$.

4. Consider an irreducible birth death chain on the nonnegative integers such that

$$\frac{q_x}{p_x} = \left(\frac{x}{x+1}\right)^2, \quad x \geq 1.$$

   (a) Show that this chain is transient.

   (b) Find $\rho_{x0}, x \geq 1$.

5. Let $X_n, n \geq 1$ be a branching chain and suppose that the associated random variable ξ has mean μ and finite variance σ^2.

   (a) Show that

$$\mathbb{E}\left[X_{n+1}^2 | X_n = x\right] = x\sigma^2 + x^2\mu^2$$

   and that

$$\mathbb{E}\left[X_n^2 | X_0 = x\right] = x\sigma^2\left(\mu^{n-1} + \cdots + \mu^{2(n-1)}\right), \quad n \geq 1.$$

   (b) Show that if there are x particles initially, then for $n \geq 1$,

$$\mathrm{Var}(X_n) = \begin{cases} x\sigma^2\mu^{n-1}\left(\frac{1-\mu^n}{1-\mu}\right), & \mu \neq 1, \\ nx\sigma^2, & \mu = 1. \end{cases}$$

1.(a) $p_x = 9/19 \quad q_x = 10/19$

$x = 1000, \quad a = 0, \quad b = 1001$

$u(x) = P(T_a < T_b \mid X_0 = x)$

From notes:

$$u(x) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y} \qquad a < x < b$$

$$\gamma_y := \prod_{j=1}^{y} (q_j/p_j) \quad y \geq 1; \; \gamma_0 = 1$$

(b) Expected ~~loss~~

$= 1000 \times u(x) - 1 \times (1 - u(x))$

2. From class we know that

$$P(T_0 < T_n \mid X_0 = x)$$

$$= \frac{\sum_{y=x}^{n-1} \gamma_y}{\sum_{y=0}^{n-1} \gamma_y}, \quad 0 < x < n$$

$$= 1 - \frac{\sum_{y=0}^{x-1} \gamma_y}{\sum_{y=0}^{n-1} \gamma_y}, \quad 0 < x < n$$

where

$$\gamma_y := \frac{q_1}{p_1} \frac{q_2}{p_2} \cdots \frac{q_y}{p_y} \qquad y > 0.$$

Noticing that

$$1 \leq T_{x+1} < T_{x+2} < T_{x+3} < \cdots$$

we see that

$$\left(T_0 < T_n \,\middle|\, X_0 = x\right) \uparrow \quad \text{for} \quad n \geq x+1$$

and hence

$$P\left(T_0 < \infty \,\middle|\, X_0 = x\right) = \lim_n P\left(T_0 < T_n \,\middle|\, X_0 = x\right)$$

$$= 1 - \frac{\sum_{y=0}^{x-1} \gamma_y}{\sum_{y=0}^{\infty} \gamma_y} \quad \text{——(1)}$$

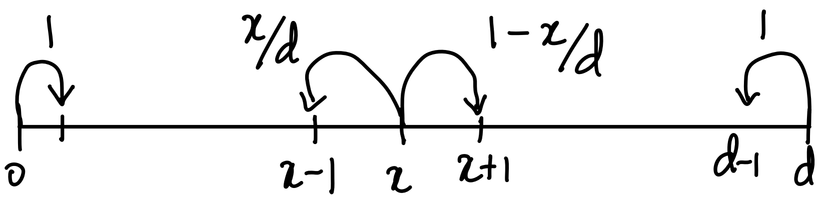
From (1), we see that (a) and (b) hold.

3. $X_n :=$ number of balls in Box I after n trials

$d - X_n :=$ number of balls in Box II after n trials

We see that $\{X_n, n \geq 1\}$ is a Markov chain with state space $\{0, 1, 2, \ldots, d\}$ and transition function

$$P(x, y) = \begin{cases} x/d & y = x-1 \\ 1 - x/d & y = x+1 \\ 0 & \text{o.w.} \end{cases}$$



Now notice that

$$
\begin{aligned}
\mathbb{E}\left[X_{n+1} \mid X_0 = x\right]
&= 1 + \left(1 - \frac{2}{d}\right)\mathbb{E}\left[X_n \mid X_0 = x\right] \\
&= 1 + \left(1 - \frac{2}{d}\right) + \left(1 - \frac{2}{d}\right)^2 \mathbb{E}\left[X_{n-1} \mid X_0 = x\right] \\
&\ \ \vdots \\
&= 1 + \sum_{j=1}^{n-1}\left(1 - 2/d\right)^j + \left(1 - \frac{2}{d}\right)^n x .
\end{aligned}
$$

4. Use the answer from Problem 2.

5. (a)

$$\mathbb{E}\left[X_{n+1}^2 \,\middle|\, X_n = x\right] = \mathbb{E}\left[\left(\xi_1 + \xi_2 + \cdots + \xi_x\right)^2\right]$$

where ξ_j, $j = 1, 2, \ldots, x$ are iid with mean μ and variance σ^2.

Therefore,

$$\mathbb{E}\left[X_{n+1}^2 \,\middle|\, X_n = x\right] = x\left(\mu^2 + \sigma^2\right) + 2\frac{x(x-1)}{2}\mu^2$$

$$= x\sigma^2 + x^2\mu^2$$

Use above for other parts.