PRACTICE PROBLEMS, STAT 532 (Pasupathy), Fall 2022

1. A Markov chain has the state space $\{0, 1, 2\}$ and transition matrix

$$\begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}.$$

   Show that the chain has a unique stationary distribution π. Find π.

2. Consider a Markov chain having transition function P such that $P(x, y) = \alpha_y, x \in \mathcal{S}$. Show that the chain has a unique stationary distribution given by

$$\pi(y) = \alpha_y, \quad y \in \mathcal{S}.$$

3. Let π be a stationary distribution of a Markov chain. Show that if $\pi(x) > 0$ and x leads to y, then $\pi(y) > 0$.

4. Let π be a stationary distribution of a Markov chain. Suppose y and z are two states such that for some constant c,

$$P(x, y) = cP(x, z), \quad x \in \mathcal{S}.$$

   Show that $\pi(y) = c\pi(z)$.

5. Let π_0 and π_1 be distinct stationary distributions of a Markov chain.

   (a) Show that for $0 \leq \alpha \leq 1$, the function

$$\pi_\alpha(x) = (1 - \alpha)\pi_0(x) + \alpha\pi_1(x), \quad x \in \mathcal{S}$$

   is a stationary distribution.

   (b) Show that the distinct values of α determine distinct stationary distributions.

6. Consider a birth and death chain on the nonnegative integers and suppose that $p_0 = 1, p_x = p > 0$ for $x \geq 1$, and $q_x = q = 1-p > 0$ for $x \geq 1$. Find the stationary distribution when it exists.

7. Consider a Markov chain having state space $\{0, 1, \ldots, 6\}$ and transition matrix

$$\begin{bmatrix} 1/2 & 0 & 1/8 & 1/4 & 1/8 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

(a) Find the irreducible closed sets of recurrent states, and stationary distribution concentrated on each of these sets.

(b) Find $\lim_n G_n(x, y)/n$.

8. A particle moves according to a Markov chain on $\{1, 2, \ldots, c + d\}$, where c and d are positive integers. Starting from any one of the first c states, the particle jumps in one transition to a state chosen uniformly from the last d states; starting from any of the last d states, the particle jumps in one transition to a state chosen uniformly from the first c states.

   (a) Show that the chain is irreducible.

   (b) Find the stationary distribution.

9. Consider a Markov chain on the nonnegative integers having transition function P given by $P(x, x + 1) = p$ and $P(x, 0) = 1 - p$, where $0 < p < 1$. Show that this chain has a unique stationary distribution π and find π.

10. The transition function of a Markov chain is called *doubly stochastic* if $\sum_{x \in \mathcal{S}} P(x, y) = 1, y \in \mathcal{S}$. What is the stationary distribution of an irreducible Markov chain having $d < \infty$ states and a doubly stochastic transition function?

1. Explicitly solve for π.

2. $P(x,y) = \alpha_y$

$$P^2(x,y) = \sum_z P(x,z)P(z,y)$$

$$= \sum_z \alpha_z \alpha_y = \alpha_y .$$

$\vdots$

$$P^n(x,y) = \alpha_y$$

$$\Rightarrow \lim_n P^n(x,y) = \alpha_y$$

Also, $\pi(y) = \alpha_y$ is stationary since

$$\sum_x \alpha_z \alpha_y = \alpha_y .$$

Conditions for uniqueness are satisfied.

3. If $x \to y$, then $\exists\, n \geq 1$ such that $P^n(x, y) > 0$. Moreover, since π is stationary,

$$\sum_z \pi(z)\, P^n(z, y) = \pi(y)$$

In the summation above, we see that for $z = x$, $\pi(z) > 0$, $P^n(z, y) > 0$.

$\boxtimes$

4. $P(x, y) = c\, P(x, z) \qquad \forall x \in S$

$$\sum_x \pi(x)\, P(x, y) = \pi(y) \quad \text{——} (1)$$

$$\sum_x \pi(x)\, P(x, z) = \pi(z) \quad \text{——} (2)$$

Plug $P(x,y) = c\,P(x,z)$ in (1)

to see that $\pi(y) = c\,\pi(z)$

$\boxed{/\!/}$

5. We see that

$$\sum_x \pi_\alpha(x) P(x,y)$$

$$= \alpha \sum_x \pi_0(x) P(x,y) + (1-\alpha) \sum_x \pi_1(x) P(x,y)$$

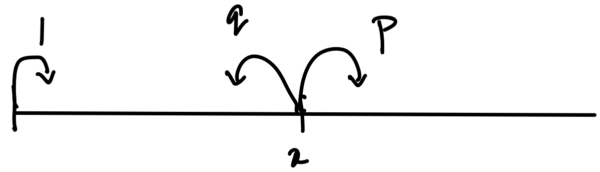
$$= \alpha\,\pi_0(y) + (1-\alpha)\,\pi_1(y).$$

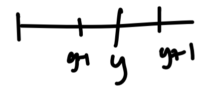
$$= \pi_\alpha(y).$$

If π_0, π_1 are distinct, each π_α is clearly distinct.

$\boxed{/\!/}$

6.





$$\pi(y-1)\,p + \pi(y+1)\,q = \pi(y) \qquad y \geq 1 \qquad \text{— (1)}$$

$$\pi(1)\,q = \pi(0) \qquad \text{— (2)}$$

$$\Rightarrow \quad \pi(y+1) - \pi(y) = \frac{p}{q}\left(\pi(y) - \pi(y-1)\right) \qquad y \geq 1 \qquad \text{— (3)}$$

$$\Rightarrow \quad \pi(y+1) - \pi(y) = \left(\frac{p}{q}\right)^{y}\left(\pi(1) - \pi(0)\right)$$

$$= \left(\frac{p}{q}\right)^{y}\left(\frac{1}{q} - 1\right)\pi(0) \qquad y \geq 1$$

Summing for $y = 0, 1, \dots y$

$$\pi(y+1) - \pi(0) = \left(\frac{1}{q} - 1\right) \pi(0) \sum_{x=0}^{y} (p/q)^{x}$$

$$\Rightarrow \quad \pi(y+1) = \pi(0)\left(1 + \left(\frac{1}{q} - 1\right) \sum_{x=0}^{y} (p/q)^{x}\right).$$

$y \geq 1$

——— (4)

Use (4), (2) and $\sum_{y=0}^{\infty} \pi(y) = 1$ to solve.

$\boxed{\checkmark}$

7. By observation, $\{0\}$ is a transient set and $\{1, 2, 3\}$, $\{4, 5, 6\}$ are irreducible closed sets of recurrent states.

Stationary distbn. concentrated on $\{1, 2, 3\}$ is $\{0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0\}$.

Stationary distbn concentrated on $\{4, 5, 6\}$ is $\{0, 0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$.

$\boxtimes$

8. (a) Notice that if $x \in \{1, 2, \ldots, c\}$

and $y \in \{c+1, c+2, \ldots, c+d\}$

$$P(x, y) > 0. \qquad \text{——} (1)$$

And if $x \in \{c+1, c+2, \ldots, c+d\}$

and $y \in \{1, 2, \ldots, c\}$

$$P(x, y) > 0 \qquad \text{——} (2)$$

Hence, if $x, y \in \{1, 2, \ldots, c\}$ or

$x, y \in \{c+1, c+2, \ldots, c+d\}$,

$$P^2(x, y) > 0.$$

(b) $$\sum_{x=1}^{c} \pi(x) \times \frac{1}{d} = \pi(y) \quad y = c+1, c+2, \dots c+d.$$

$$\sum_{x=c+1}^{c+d} \pi(x) \times \frac{1}{c} = \pi(y) \quad y = 1, 2, \dots c$$

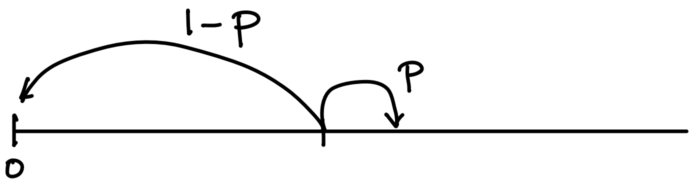
Solve to get

$$\pi(x) = \frac{d}{c} \frac{1}{c+d} \quad x = 1, 2, \dots, c$$

$$= \frac{c}{d} \frac{1}{c+d} \quad x = c+1, c+2, \dots, c+d.$$

$\square$

9.



$$P(x,y) = \begin{cases} p & x+1 \\ 1-p & 0 \\ 0 & \text{ow} \end{cases}$$

Lets try to solve for π.

$$\pi(y-1)\, p = \pi(y) \qquad y \neq 0$$

$$\sum_{x=0}^{\infty} (1-p)\, \pi(x) = \pi(0)$$

$$\Rightarrow \quad 1 = \frac{\pi(0)}{1-p} \quad \Rightarrow \quad \pi(0) = 1-p.$$

and

$$\pi(y) = \pi(0)\, p^y = (1-p)\, p^y \;. \; y \neq 0$$

$\square$

10. Let the Markov chain have ~~s~~states $1, 2, \ldots, d$.

By observation,

$$\pi(x) = \frac{1}{d}, \quad x = 1, 2, \ldots, d.$$