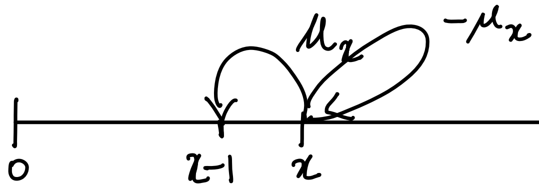
ASSIGNMENT VIII, STAT 532 (Pasupathy)

1. Consider a pure death process on $\{0, 1, 2 \ldots, \}$.
   (a) Write the forward equation.
   (b) Find $P_{xx}(t)$.
   (c) Solve for $P_{xy}(t)$ in terms of $P_{x,y+1}(t)$.
   (d) Find $P_{x,x-1}(t)$.
   (e) Show that if $\mu_x = x\mu, x \geq 0$, for some constant μ, then
   $$P_{xy}(t) = \binom{x}{y} (e^{-\mu t})^y (1 - e^{-\mu t})^{x-y}, \quad 0 \leq y \leq x.$$
2. Let $X(t), t \geq 0$ be the infinite server queue and suppose that initially there are x customers present. Compute the mean and variance of $X(t)$.
3. Consider a birth death process $X(t), t \geq 0$ with state space $\{0, 1, 2, \ldots, \}$ with birth and death rates
   $$\lambda_x = x\lambda \text{ and } \mu_x = x\mu, x \geq 0,$$
   where λ and μ are nonnegative integers. Show that $\mathbb{E}[X(t) \,|\, X(0) = x] = xe^{(\lambda-\mu)t}$.
4. In the previous problem, find $\mathrm{Var}(X(t) \,|\, X(0) = x)$.
5. Consider a positive recurrent irreducible birth and death process on $\{0, 1, 2, \ldots, \}$, and let $X(0)$ have the stationary distribution π for its initial distribution. Then $X(t)$ has distribution π for all $t \geq 0$. The quantities $\sum_{x=0}^{\infty} \lambda_x \pi(x)$ and $\sum_{x=0}^{\infty} \mu_x \pi(x)$ are the average birth rate and the average death rate of the process, respectively. Show that these quantities are equal.
6. Suppose d particles are distributed into two boxes. A particle in box 0 remains in that box for a random length of time that is exponentially distributed with parameter λ before going into box 1. A particle in box 1 remains there for an amount of time that is exponentially distributed with parameter μ before going into box 0. The particles act independently of each other. Let $X(t)$ denote the number of particles in box 1 at time $t \geq 0$. Notice that is a birth and death process.
   (a) Find the birth and death rates.
   (b) Find $P_{xd}(t)$.
   (c) Find $\mathbb{E}[X(t) \,|\, X(0) = x]$.
   (d) Find the stationary distribution.

1.



, $\mu_0 = 0$

(a)

$$P'_{xy}(t) = P_{x,y+1}(t)\,\mu_{y+1} - P_{x,y}(t)\,\mu_y$$

Note that $P_{x,y+1}(t) = 0 \quad \forall\ y \geq x$

(b)

$$P'_{xx}(t) = -\mu_x P_{x,x}(t)$$

$$\Rightarrow \log P_{x,x}(t) = -\mu_x t$$

$$\Rightarrow P_{x,x}(t) = e^{-\mu_x t}$$

(c) From (a), for $y \neq x$

$$P'_{xy}(t) + \mu_y P_{xy}(t) = \mu_{y+1} P_{x,y+1}(t)$$

$$\Rightarrow e^{\mu_y t} P'_{xy}(t) + \mu_y e^{\mu_y t} P_{xy}(t)$$
$$= \mu_{y+1} P_{x,y+1}(t)\, e^{\mu_y t}$$

$$\Rightarrow e^{\mu_y t} P_{xy}(t) = \mu_{y+1} \int_0^t P_{x,y+1}(s)\, e^{\mu_y s}\, ds$$

$$P_{xy}(t) = \mu_{y+1} \int_0^t e^{-\mu_y (t-s)} P_{x,y+1}(s)\, ds$$

$y < x,\ t \geq 0$

(d) From (c),

$$P_{x,x-1}(t) = \mu_x \int_0^t e^{-\mu_x(t-s)} e^{-\mu_x s} ds$$

$$= \mu_x t e^{-\mu_x t}.$$

$\left(P(N(t)=1) \text{ where } N(t) \sim \text{Poisson}(\mu_x t) \right)$

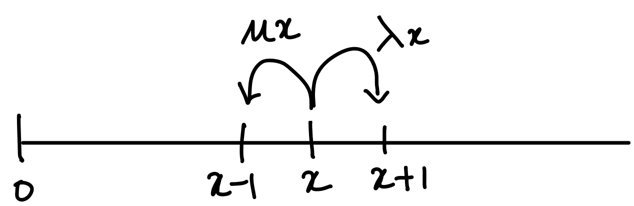
(e) Use the results (a) – (d) and integrate explicitly.

2. We know that

$$X(t) \sim \text{Poisson}\left(\frac{\lambda}{\mu}\left(1-e^{-\mu t}\right)\right)$$

$$\text{Mean} = \text{Variance} = \frac{\lambda}{\mu}\left(1-e^{-\mu t}\right) t.$$

3.



$$q_x = x(\lambda+\mu)$$

0 is an absorbing state.

$$P'_{xy}(t) = \begin{cases} (y-1)P_{x,y-1}(t)\lambda - yP_{x,y}(t)(\lambda+\mu) \\ \qquad + (y+1)P_{x,y+1}(t)\mu & y \geq 1 \\ P_{x,1}(t)\mu & y = 0 \end{cases}$$

$$\Rightarrow \sum_{y=0}^{\infty} y P'_{xy}(t) = \sum_{y=1}^{\infty} y(y-1) P_{x,y-1}(t)\lambda$$

$$- \sum_{y=1}^{\infty} y^2 P_{x,y}(t)(\lambda+\mu)$$

$$+ \sum_{y=1}^{\infty} y(y+1) P_{x,y+1}(t)\mu$$

$$= \sum_{y=0}^{\infty} \Big((y+1) y \lambda - y^2 (\lambda+\mu) + (y-1) y \mu \Big) P_{x,y}(t)$$

$$= \sum_{y=0}^{\infty} (\lambda-\mu)\, y\, P_{x,y}(t) \qquad \text{——— (1)}$$

Denoting $m_x(t) = \mathbb{E}\left[X(t) \mid X(0)=x \right]$

(1) gives

$$m_x'(t) = (\lambda-\mu)\, m_x(t), \quad t \geq 0$$

Integrate fom o to t to get

$$\ln m_x(t)/m_x(0) = (\lambda-\mu)t.$$

$$\Rightarrow \quad m_x(t) = x e^{(\lambda-\mu)t}, \quad t \geq 0.$$

4.

$$P'_{xy}(t) = \begin{cases} (y-1) P_{x,y-1}(t)\lambda - y P_{x,y}(t)(\lambda+\mu) \\ \qquad + (y+1) P_{x,y+1}(t)\mu & y \geq 1 \\ P_{x,1}(t)\mu & y=0 \end{cases}$$

Again, we use forward equation above to get

$$\sum_{y=0}^{\infty} y^2 P'_{xy}(t) = \sum_{y=1}^{\infty} y^2 (y-1) P_{x,y-1}(t)\lambda$$

$$- \sum_{y=1}^{\infty} y^3 P_{x,y}(t)(\lambda+\mu)$$

$$+ \sum_{y=1}^{\infty} y^2 (y+1) P_{x,y+1}(t)\mu$$

$$= \sum_{y=0}^{\infty} \left((y+1)^2 y \lambda - y^3(\lambda+\mu) + (y-1)^2 y \mu \right) P_{xy}(t)$$

$$= \sum_{y=0}^{\infty} 2y^2(\lambda-\mu) + y(\lambda+\mu) P_{xy}(t)$$

Denote $v_x(t) = \mathbb{E}\left[X^2(t) \mid X(0)=x \right]$

$$v_x'(t) = 2(\lambda-\mu) v_x(t) + (\lambda+\mu) m_x(t)$$

$t \geqslant 0.$

$$\Rightarrow e^{-2(\lambda-\mu)t} v_x'(t) = e^{-2(\lambda-\mu)t} 2(\lambda-\mu) v_x(t)$$
$$+ e^{-2(\lambda-\mu)t} (\lambda+\mu) m_x(t)$$

Solving, we get

$$V_x(t) = x^2 e^{2(\lambda-\mu)t} + x\frac{(\lambda+\mu)}{(\lambda-\mu)}\left(e^{2(\lambda-\mu)t} - e^{(\lambda-\mu)t}\right)$$

$$\mathrm{Var}\left(X(t) \mid X(0) = x\right)$$

$$= V_x(t) - m_x^2(t)$$

$$= x\frac{(\lambda+\mu)}{(\lambda-\mu)}\left(e^{2(\lambda-\mu)t} - e^{(\lambda-\mu)t}\right).$$

5. The stationary equation is

$$\pi(1)\mu_1 - \pi(0)\lambda_0 = 0$$

$$\pi(y+1)\mu_{y+1} - \pi(y)\lambda_y = \pi(y)\mu_y - \pi(y-1)\mu_{y-1}$$

$$y \geq 1$$

We can guess that the solution satisfies

$$\pi(y+1)\mu_{y+1} = \pi(y)\lambda_y \quad y \geq 0.$$

Sum for $y = 0, 1, 2, \ldots$ to see that

$$\sum_{y=0}^{\infty} \pi(y)\mu_y = \sum_{y=0}^{\infty} \pi(y)\lambda_y$$

after recalling that $\mu_0 = 0$.