Recap from Wednesday

• Rationale for CAMFR
• Software architecture
• Basic Applications
  – 1D waveguides
  – 2D waveguides
  – 3D cylindrical waveguides
• Advanced Applications
  – Photonic Crystal Splitters
  – VCSELs
Outline

• Recap from Wednesday
• Overview of Coupled Mode Theory
• Derivation of Coupled Mode Equations
• Applications:
  – Single Waveguides
  – Add-Drop filters
  – Waveguide Bends
  – Channel Drop
  – T-Splitters
  – Nonlinear Kerr Waveguides
Coupled-Mode Theory: Basic Concept

Energy exists in 2 forms:
- Localized resonant modes: \( \{A_i\} \)
- Traveling waveguide modes: \( \{S_{i+}, S_{i-}\} \)

Key assumptions:
- Weak coupling between modes
- Linearity (i.e., the validity of superposition)
- Time-reversal symmetry and conservation of energy
- Time-invariance

Derivation of Coupled Mode Equations

• Assume that:
  – Energy of resonant modes is given by \( U_i = |A_i|^2 \)
  – Incident power of waveguide modes is given by \( |S_{i+}|^2 \)
• Resonator \( i \) oscillates in phase at frequency \( \omega_i \), hence:
  \[
  \frac{dA_i}{dt} = -j \omega_i A_i
  \]
• Resonator energy decays at rate proportional to energy present:
  \[
  \frac{dU_i}{dt} = -\frac{2U_i}{\tau_i}
  \]
  \[
  \frac{dA_i}{dt} = -j \omega_i A_i - \frac{A_i}{\tau_i}
  \]
Derivation of Coupled Mode Equations

• By linearity, coupling of waveguides into modes given by:

\[
\frac{dA_i}{dt} = \cdots + \sum_j \alpha_{ij} S_{j+}
\]

• For similar reasons, outgoing waveguide modes given by:

\[
S_{i-} = \beta_i S_{i+} + \sum_j \gamma_{ij} A_j
\]

• By conservation of energy, inputs must be stored or lost:

\[
\sum_i \left[ |S_{i+}|^2 - |S_{i-}|^2 - \frac{dU_i}{dt} \right] = 0
\]

• Special cases can be used to obtain coefficients: \(\{\alpha_{ij}, \beta_i, \gamma_{ij}\}\)
Derivation of Coupled Mode Equations

• In absence of coupling to resonant modes, conservation of energy requires $|\beta_i| = 1$. Phase depends on convention.
• In absence of input waveguide, must have:

\[
0 = |S_{i-}|^2 + \frac{dU_i}{dt}
\]

\[
0 = |S_{i-}|^2 - \frac{2U_i}{\tau_i}
\]

\[
0 = |\gamma_i|^2 U_i - \frac{2U_i}{\tau_i}
\]

• Thus, $\gamma_i = \sqrt{2/\tau_i}$
• Finally, time reversal implies $\alpha_i = \gamma_i$
Single Waveguide

- For simplest case: 1 waveguide + 1 resonator with 1 input:

\[ S_{1-} = S_{1+} - \sqrt{2/\tau_1}A \]

\[ \frac{dA}{dt} = -j\omega_o A - A \frac{A}{\tau_1} + \sqrt{\frac{2}{\tau_1}} S_{1+} \]

- Transmission can be calculated as quotient:

\[
R(\omega) = \frac{|S_{1-}|^2}{|S_{1+}|^2} = \frac{(\omega - \omega_o)^2 + \tau_1^{-2}}{\omega - \omega_o)^2 + \tau_1^{-2}}
\]

- Result: full reflection at all wavelengths, since light has no where else to go!
Application: Add-Drop Filters

- For simple case: 2 waveguides + 1 resonator with 1 input:
  \[ S_{1-} = S_{1+} - \sqrt{2/\tau_1}A \]
  \[ S_{2-} = \sqrt{2/\tau_2}A \]
  \[ \frac{dA}{dt} = -j\omega_0 A - \frac{A}{\tau_1} - \frac{A}{\tau_2} + \sqrt{\frac{2}{\tau_1}}S_{1+} \]

- Transmission can be calculated as quotient:
  \[ T(\omega) = \frac{|S_{1-}|^2}{|S_{1+}|^2} = \frac{(\omega - \omega_o)^2 + (\tau_1^{-1} - \tau_2^{-1})^2}{(\omega - \omega_o)^2 + (\tau_1^{-1} + \tau_2^{-1})^2} \]

- Result: a Lorentzian dip in transmission, centered at resonant frequency \( \omega_o \)
Application: Waveguide Bends

- Can understand a photonic waveguide bend as a special case of the previous problem, with outputs reversed.

J.D. Joannopoulos et al., *Photonic Crystals*, Ch. 10 (Princeton, 2008)
Application: Channel Drop Filter

- Consider 4 channels + 2 resonators with 1 input:
  \[ S_{1-} = S_{1+} - \sqrt{2/\tau_1} A_1 - \sqrt{2/\tau_2} A_2 \]
  \[ S_{2-} = \sqrt{2/\tau_1} A_1 + \sqrt{2/\tau_2} A_2 \]
  \[ S_{34-} = \sqrt{2/\tau_3} A_1 + \sqrt{2/\tau_4} A_2 \]
  \[ \frac{dA_1}{dt} = -j\omega_1 A_1 - \sum_i \frac{A_1}{\tau_i} + \sqrt{\frac{2}{\tau_1}} S_{1+} \]

- Transmission can be calculated as quotient:
  \[ T(\omega) = \frac{|S_{1-}|^2}{|S_{1+}|^2} = \left| 1 - \frac{2/\tau_1}{j(\omega_1 - \omega) + \Gamma} - \frac{2/\tau_2}{j(\omega_2 - \omega) + \Gamma} \right|^2 \]

- Result: a Fano lineshape encompassing both resonances \( \omega_1 \) and \( \omega_2 \)
Application: Channel Drop Filter

- Can tune resonator pair to transmit into any desired channel at a target frequency

J.D. Joannopoulos et al., *Photonic Crystals*, Ch. 10 (Princeton, 2008)
Application: T-Splitter

- Can predict coupling strengths needed for perfect forward transmission: $\tau_1^{-1} = \tau_2^{-1} + \tau_3^{-1}$

J.D. Joannopoulos et al., *Photonic Crystals*, Ch. 10 (Princeton, 2008)
Application: Kerr Nonlinearities

• Take 2 waveguides + 1 resonator with Kerr nonlinearity and 1 input:

\[
S_{1-} = S_{1+} - \sqrt{2/\tau_1}A \\
S_{2-} = \sqrt{2/\tau_2}A
\]

\[
\frac{dA}{dt} = -j(\omega_0 + \kappa |A|^2)A - \frac{A}{\tau_1} - \frac{A}{\tau_2} + \sqrt{\frac{2}{\tau_1}}S_{1+}
\]

• System becomes bistable, with different initial conditions giving rise to different transmission regimes

• Transmission should now be calculated as:

\[
T(\omega) = \frac{2}{\tau_2} |A|^2 = \cdots = \frac{1}{1 + (\delta - P_{out}/P_b)^2}
\]

J.D. Joannopoulos et al., *Photonic Crystals*, Ch. 10 (Princeton, 2008)
Conclusions

• In general, CMT works for a broad range of systems with well-defined and relatively weakly coupled resonances
• Can be readily extended to cases with weak losses, by treating them as additional ‘waveguides’
• Furthermore, in the linear case, most problems can be solved analytically
• Can extend CMT to nonlinear systems (e.g., Kerr media) or time-varying systems, but generally must use ODE solvers to find numerical solutions
Next Class

• Is on Monday, April 1
• Next time: we will discuss coupled mode theory tools:
  http://nanohub.org/tools/cmtcomb3/