ECE 695
Numerical Simulations
Lecture 10: Beam Propagation Method

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Outline

- Beam Propagation Method
- Nonlinear Schrodinger Equation
- Fourier space BPM
- Uniform grid BPM with PML
- Finite element formulation
- Vectorial BPM mode solver

Example: Beam Propagation

- Starting from the Helmholtz equation:

$$-\nabla^2\psi = \left(\frac{n\omega}{c}\right)^2 \psi$$

- One can assume a solution of the form:

$$\psi = \phi e^{-j\beta z}$$

- Where ϕ is slowly varying, which gives rise to:

$$-\nabla^2\phi + 2j\beta\nabla\phi = k_\perp^2\,\psi$$

Beam Propagation

- To simplify problem, drop second derivatives in z – now we can write as:

$$\frac{\partial \phi}{\partial z} = \frac{j}{2\beta} \nabla_{\perp}^{2} \phi + \frac{jk_{\perp}^{2}}{2\beta} \phi$$

- Can simplify by defining two operators:

$$U = \frac{j}{2\beta} \nabla_{\perp}^{2}$$

$$W = \frac{jk_{\perp}^{2}}{2\beta}$$

$$\frac{\partial \phi}{\partial z} = (U + W)\phi$$

Beam Propagation

- For a small z-step of size *h*, we can formally write a solution:

$$\phi(z+h) = e^{h(U+W)}\phi(z)$$

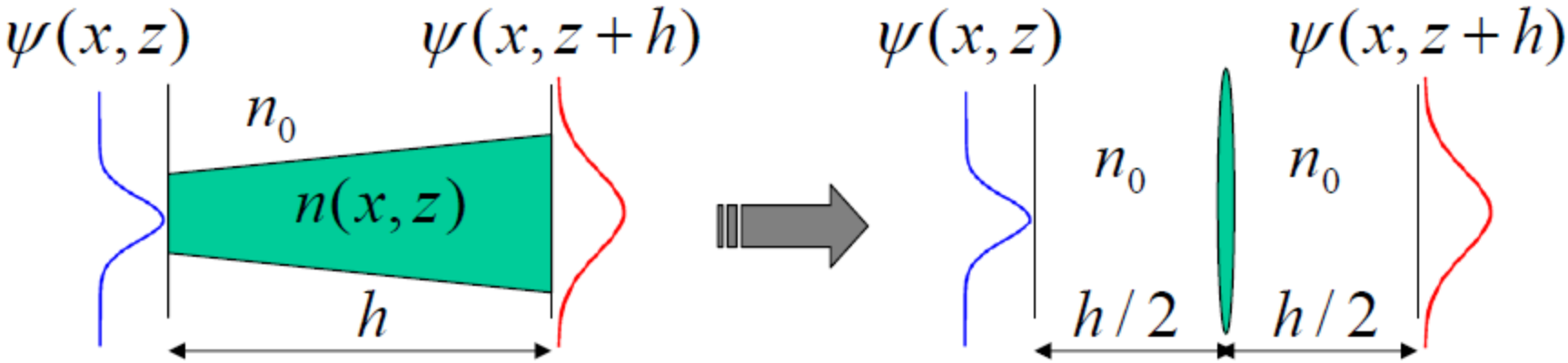
- If we know that U and W operators commute, we can rewrite as:

$$\phi(z+h) = e^{hU}e^{hW}\phi(z)$$

$$\phi(z+h) = e^{hU/2}e^{hW}e^{hU/2}\phi(z)$$

Beam Propagation

- Split-step method
  - Propagate half a step with the Laplacian
  - Propagate linear phase shift over the full distance
  - Propagate half a step with the Laplacian



Nonlinear Schrodinger Equation

- Can derive expressions suitable for understanding fibers with dispersion and Kerr nonlinearity:

$$U = -\frac{j\beta_2}{2}\frac{\partial^2}{\partial t^2}$$

$$W = -\alpha + \frac{j\kappa}{2}|\phi|^2$$

$$\frac{\partial \phi}{\partial z} = (U + W)\phi$$

Nonlinear Schrodinger Equation

- In the presence of nonlinearity, don't actually know the value of $W(z+h)$
- Can obtain the result iteratively
  - Use $W(z)$ to evaluate $W(z+h)$
  - Work backwards to refine guess for $\phi(z+h)$
- After a few iterations, generally reach a self-consistent solution

BPM Strategies

- Most important decision is handling inhomogeneity well
- Possible strategies:
  - Uniform spatial grid
  - FFT
  - Finite-element method

Uniform Spatial Grid BPM

- Reformulate Laplacian in 2D with:

$$\nabla^2\phi \approx \frac{\phi_{i-N} + \phi_{i-1} - 4\phi_i + \phi_{i+1} + \phi_{i+N}}{h^2}$$

- Where *h* is the grid spacing

FFT BPM

- Well-suited for diffraction step, where we can rephrase the operator as:

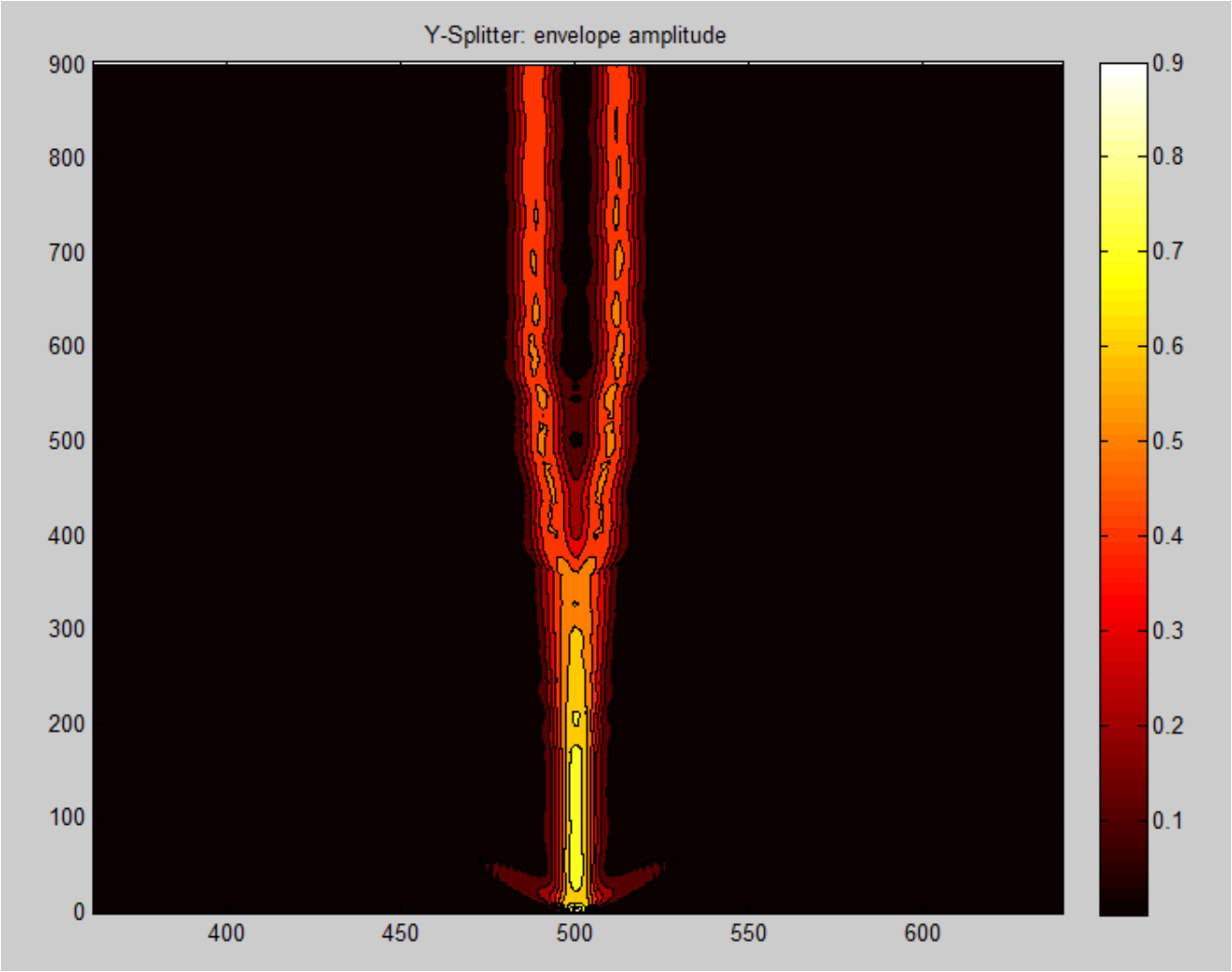
$$U = -\frac{j}{2\beta}(k + G)_{\perp}^{2}$$

- Can transform before and then back afterwards, via FFT, as in MPB solver

Example: Beam Propagation

```
[xx,yy] = meshgrid([xa:del:xb-del],[1:1:zmax]);
mode = A*exp(-((x+x0)/W0).^2);   % Gaussian pulse
dftmode = fix(fft(mode));          % DFT of Gaussian pulse
zz = imread('ybranch.bmp','BMP'); %Upload image with the profile
...
phase1 = exp((i*deltaz*kx.^2)./(nbar*k0 + sqrt(max(0,nbar^2*k0*2
- kx.^2))));
for k = 1:zmax,
    phase2 = exp(-(od + i*(n(k,:) - nbar)*k0)*deltaz);
    mode = ifft((fft(mode).*phase1)).*phase2;
    zz(k,:) = abs(mode);
end
```

Example: Beam Propagation



Perfectly Matched Layers

- In order to prevent lateral reflections (e.g., from PEC boundaries), can introduce perfectly matched layers (PML)
- Several formulations (including split-field and uniaxial), but here we'll follow stretched coordinate PML
- Effected by the transformation:

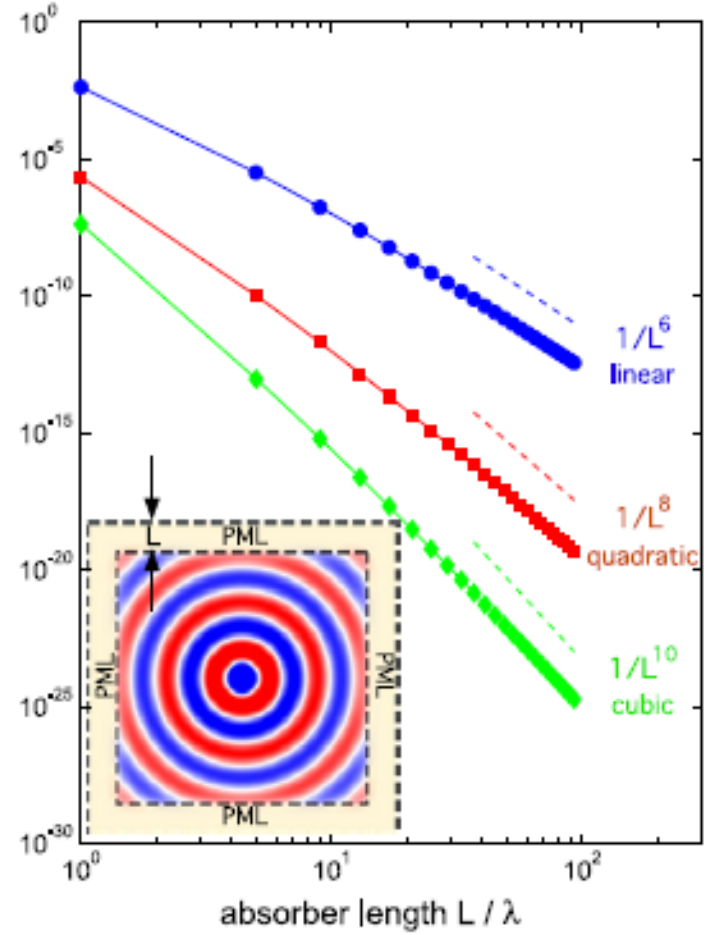
$$\nabla \longrightarrow A \cdot \nabla$$

$$A = \begin{pmatrix} 1 - j\beta & 0 & 0 \\ 0 & 1 - j\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\beta = -\frac{3\lambda\rho^2}{4\pi n d^3} \ln R$$

Perfectly Matched Layers

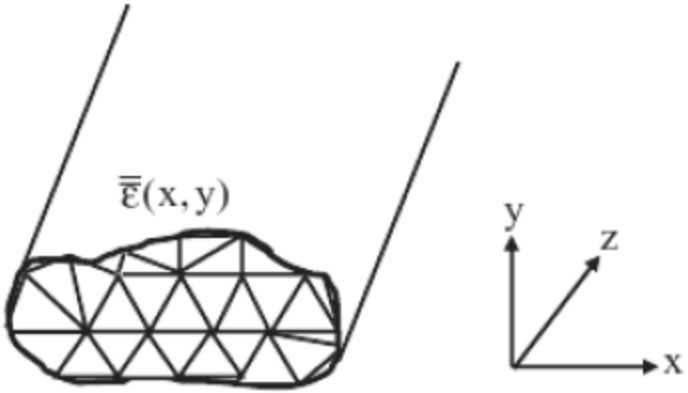
- Residual reflection scales as a power law with PML thickness
- Cubic absorption increase with position offers the best performance



A.F. Oskooi *et al.*, *Comput. Phys. Commun.* (2009)

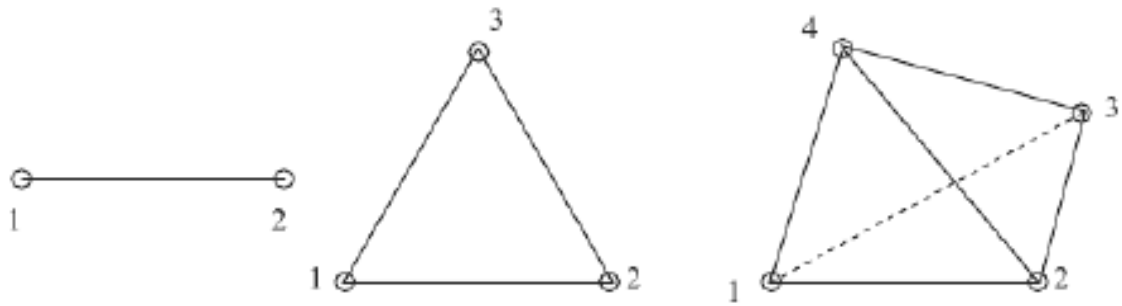
Finite Element BPM

- Finite element method consists of dividing a spatial domain in 1D, 2D or 3D into a mesh
- Mesh generally has D+1 vertices
- Solution can take various forms, but usually a tent function within each D+1-gon



Finite Elements

- Shapes: 1D, 2D, and 3D



- Shape functions:

$$\text{1D: } u(x) = \alpha + \beta x + \gamma x^2 + \cdots$$

$$\text{2D/3D: } u(x) = \sum_{k=0}^{d} [\alpha_k x^k + \beta_k y^k + \gamma_k z^k]$$

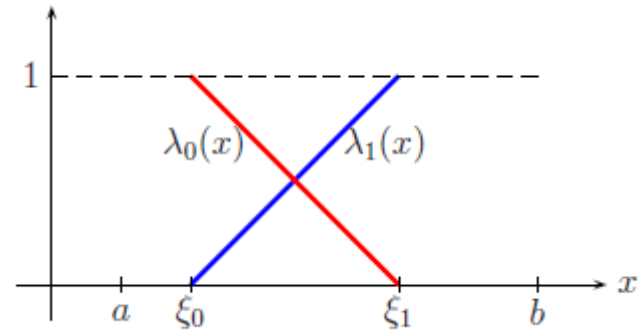
Finite Elements

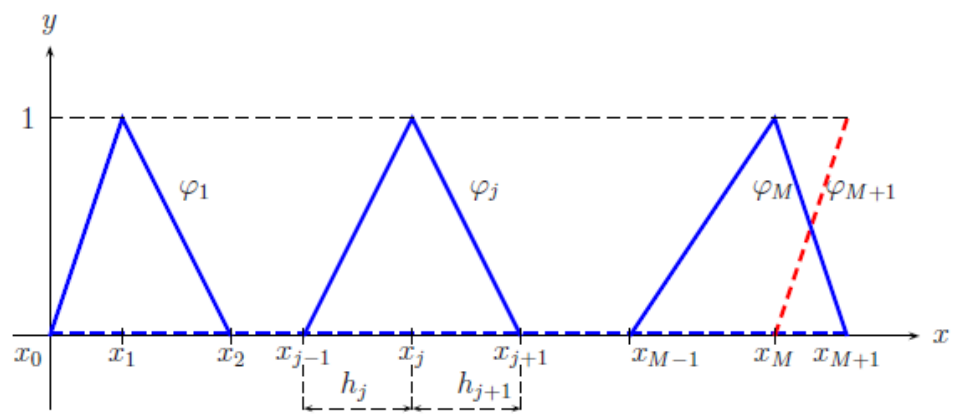
- Lagrange functions:

$$\lambda_o(x) = \frac{\xi_1 - x}{\xi_1 - \xi_o}$$

$$\lambda_1(x) = \frac{x - \xi_o}{\xi_1 - \xi_o}$$

Basis functions $\varphi_j(x)$ combine the Lagrange functions with compact support





M. Asadzadeh, *Introduction to the Finite Element Method for Differential Equations* (2010)

Finite Element BPM

- In general, can formulate FE problems as:

$$Lu = b$$

  - *L* is the stiffness matrix, representing overlap between basis functions
  - *b* is the integral of given PDE with respect to basis
  - *u* is unknown
- Value of FEA comes from:
  - *Spatial flexibility*: can define each element to vary in size quite substantially
  - *Speed*: properly chosen basis functions have compact support, leading to a sparse matrix

Finite Element BPM

- Can define error function as:

$$E = Lu - b$$

- In order to eliminate errors, set weighted residual w_i in test space *v* to zero:

$$\oint_v w_i \, (Lu - b) = 0$$

- Galerkin's method is a specific example of this:

$$\oint_v \psi(Lu - b) = 0$$

where *u*(*x*) are the polynomials we saw earlier

Finite Element BPM

- Can refine accuracy of BPM for wide-angle beam propagation with second derivative in z:

$$\frac{d\zeta}{dz} = -2j\beta\zeta - \nabla_\perp^2 \phi - k_\perp^2 \phi$$

$$\frac{d\phi}{dz} = \zeta$$

- Can then choose a <u>Padé approximant</u> based on initial value of ζ. If $\zeta(0)=0$, then:

$$\zeta = j\beta\left[\sqrt{1 + \frac{\nabla_\perp^2 + k_\perp^2}{\beta^2}} - 1\right]\phi$$

Finite Element BPM

- Applying Galerkin method to second-order BPM equations yields:

$$h_T(x, y, z) = \sum_{j=1}^{N_{px}} h_{xj}(z)\psi_j(x, y)\hat{u}_x + \sum_{j=N_{px}+1}^{N_p} h_{yj}(z)\psi_j(x, y)\hat{u}_y$$

$$[M]\frac{\partial^2\{h_T\}}{\partial z^2} - 2\gamma[M]\frac{\partial\{h_T\}}{\partial z} + ([K] + \gamma^2[M])\{h_T\} = \{0\}$$

$$[M]_{ij} = \int_{\Omega} \bar{\bar{k}}_a \vec{\psi}_j . \vec{\psi}_i d\Omega$$

$$[K]_{ij} = -\int_{\Omega} (\bar{\bar{k}}_{zz}\nabla_T \times \vec{\psi}_j).(\nabla_T \times \vec{\psi}_i)d\Omega + \int_{\Omega} (\nabla_T \times \vec{\psi}_j)\nabla_T.(\bar{\bar{k}}_b^T \vec{\psi}_i)d\Omega - \oint_{\partial\Omega} (\nabla_T.\vec{\psi}_j)(\bar{\bar{k}}_b^T \vec{\psi}_i).\hat{n}d\ell + \int_{\Omega} \bar{\bar{k}}_c \vec{\psi}_j . \vec{\psi}_i d\Omega$$

Reducing FEM Errors

- Error depends on match between true solution and basis functions
- To reduce error, can try the following:
  - H-adaptivity: decrease the mesh size
  - P-adaptivity: increase the degree of the fitted polynomials
  - HP-adaptivity: combine all of the above

Reducing FEM Errors

- Strategy for reducing errors:
  - Create an initial meshing
  - Compute solution on that meshing
  - Compute the error associated with it
  - If above our tolerance, refine the mesh spacing and start again

BPM Mode Solver

- Can extend BPM method to solve for modes, by propagating in the imaginary direction
- First, drop all derivatives in BPM equation:

$$[K]\{h_{t,l}\} = -\gamma^2[M]\{h_{t,l}\}$$

- Second, write down next step in z:

$$\{h_{t,l}\}_{k+1} = \frac{-2\gamma - 0.5\Delta z k_o^2\left(n_{eff,\ell}^2 - n_o^2\right)}{-2\gamma + 0.5\Delta z k_o^2\left(n_{eff,\ell}^2 - n_o^2\right)}\{h_{t,l}\}_k$$

- Third, substitute special value of Δz:

$$\Delta z \approx j\frac{4n_o}{\left(n_{eff,\ell}^2 - n_o^2\right)k_o}$$

BPM Mode Solver

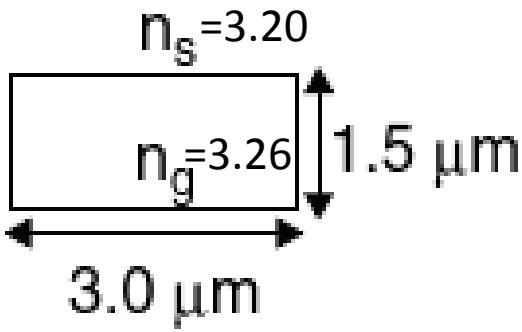
- Since Δz initially unknown, assume largest index possible, and decrease it as needed
- Will eventually converge to correct answer and effective refractive index:

$$n_{eff,\ell,k}^2 = \frac{\{h_t\}_k^*[K]_k\{h_t\}_k}{k_o^2\{h_t\}_k^*[M]_k\{h_t\}_k}$$

- Can use Gram-Schmidt normalization procedure to find higher-order modes:

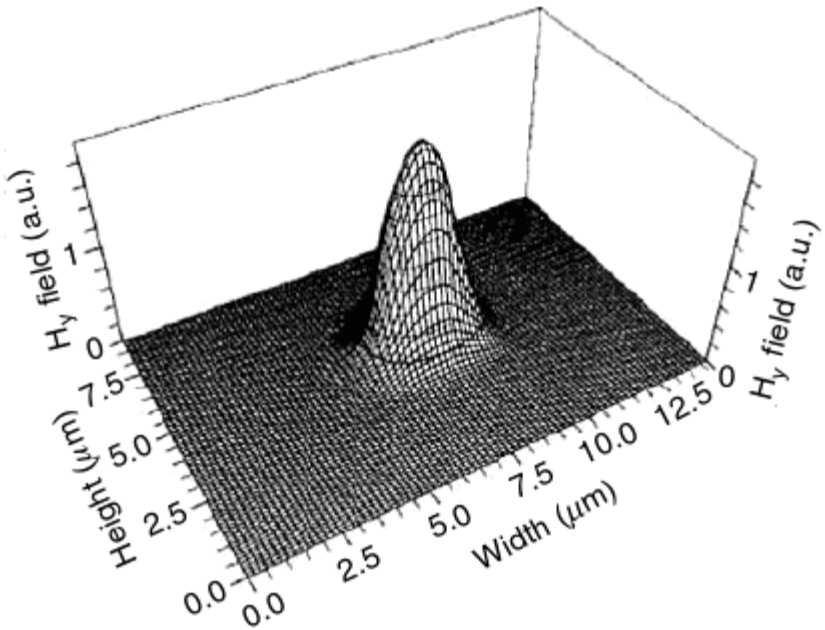
$$\{h_t\}_{1,new} = \{h_t\}_1 - \sum_{\ell=1}^{i-1} \frac{\{h_{t,l}\}^*[M]\{h_t\}_1}{\{h_{t,l}\}^*[M]\{h_{t,l}\}}\{h_{t,l}\}$$

VBPM on a Waveguide: Problem Description



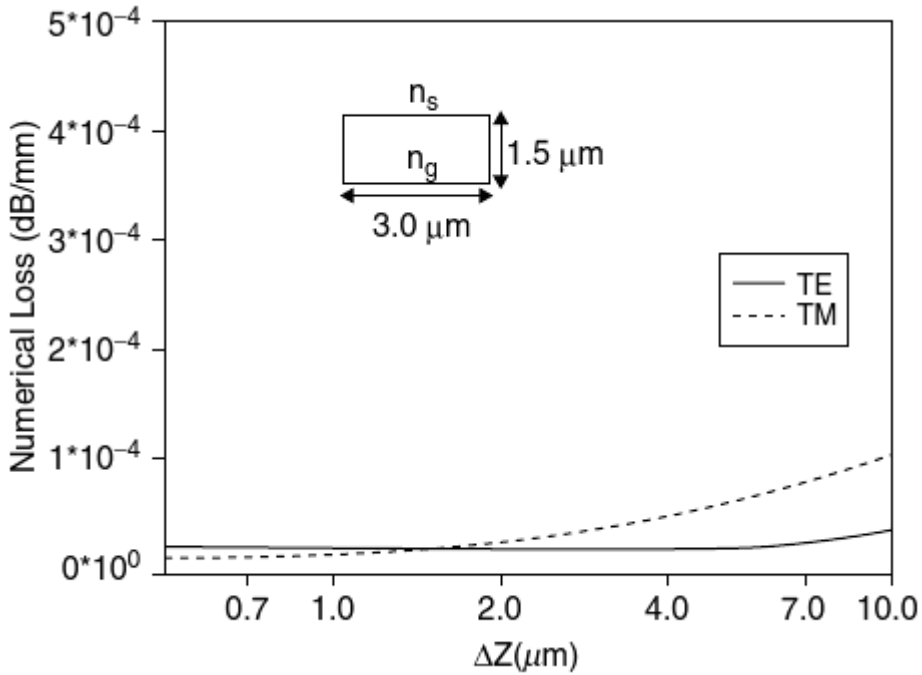
- Cross section defined above; $\lambda = 1.3\ \mu$m
- Propagation along z is semi-infinite
- Must grid space with first-order triangular elements in cross-sectional plane; choose PML to reduce reflections to 10^{-100}
- Will vary Δz for maximum effectiveness

VBPM on a Waveguide



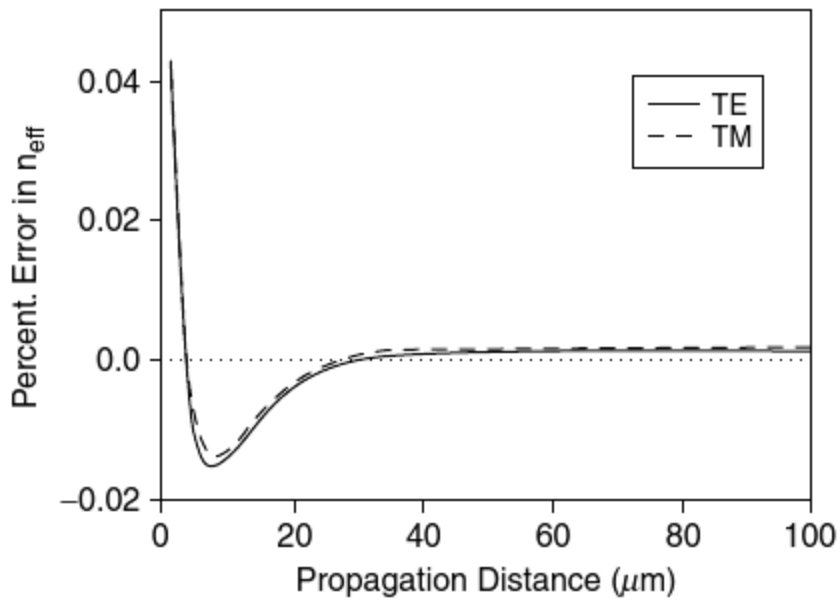
- Fundamental mode is calculated accurately with 12,800 first-order triangular elements

VBPM on a Waveguide



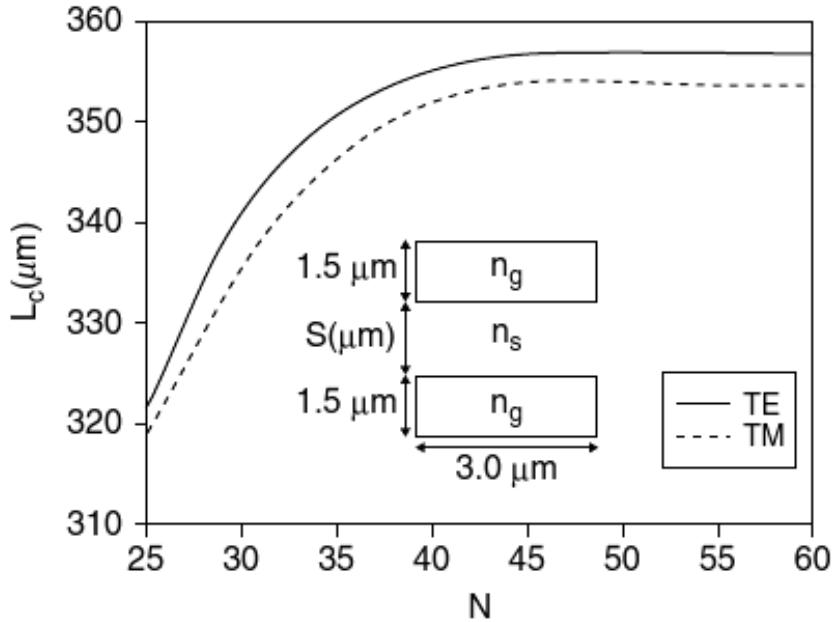
- Propagation step size in *Z*, known as ΔZ, should equal transverse dimensions for best accuracy

VBPM on a Waveguide: Longitudinal Imaginary Propagation



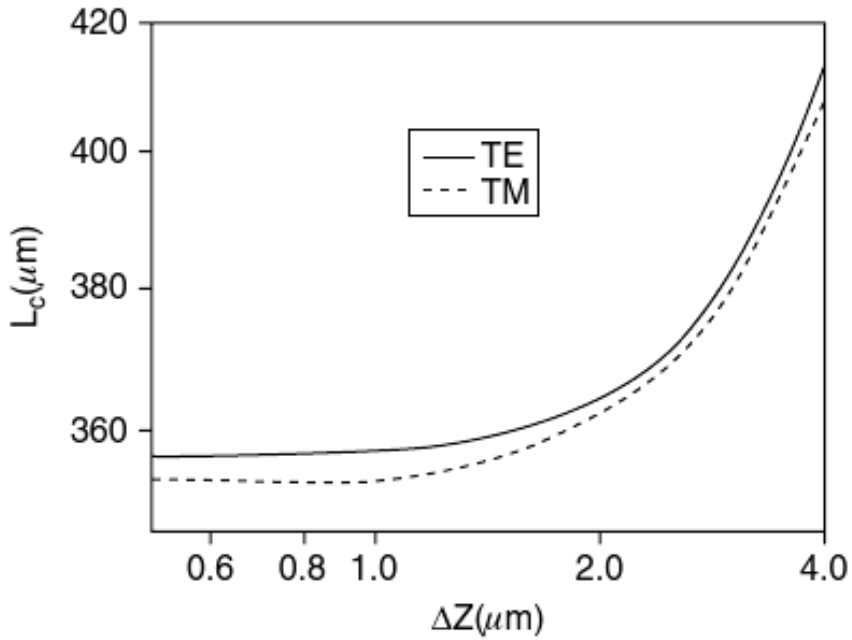
- With optimal step size, can solve the fundamental mode of both polarizations in a pretty modest number of steps!

VBPM on a Waveguide: Accuracy



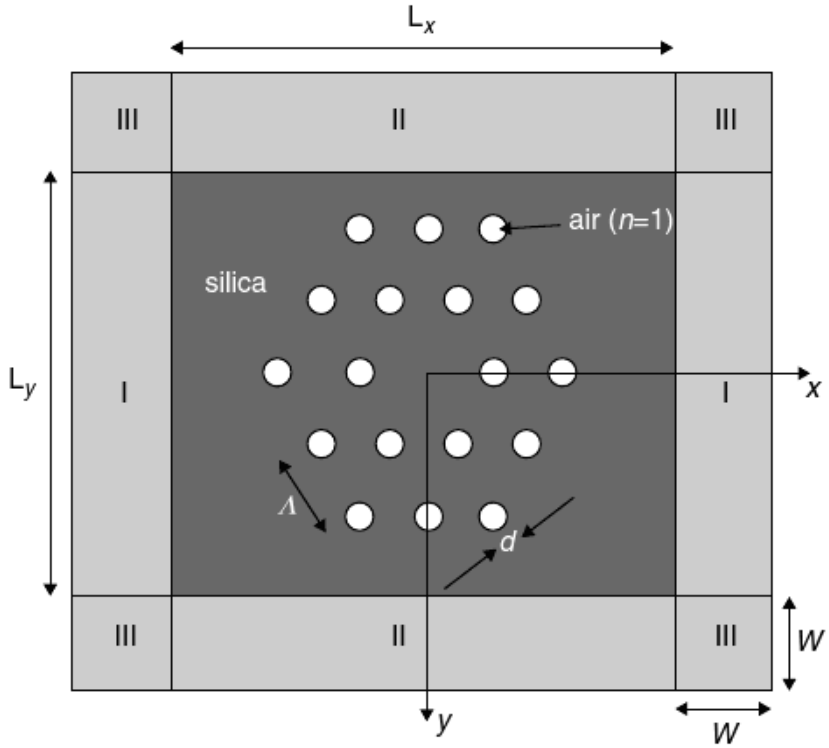
- Accuracy of calculation of waveguide coupling length as a function of mesh divisions *N*

VBPM on a Waveguide



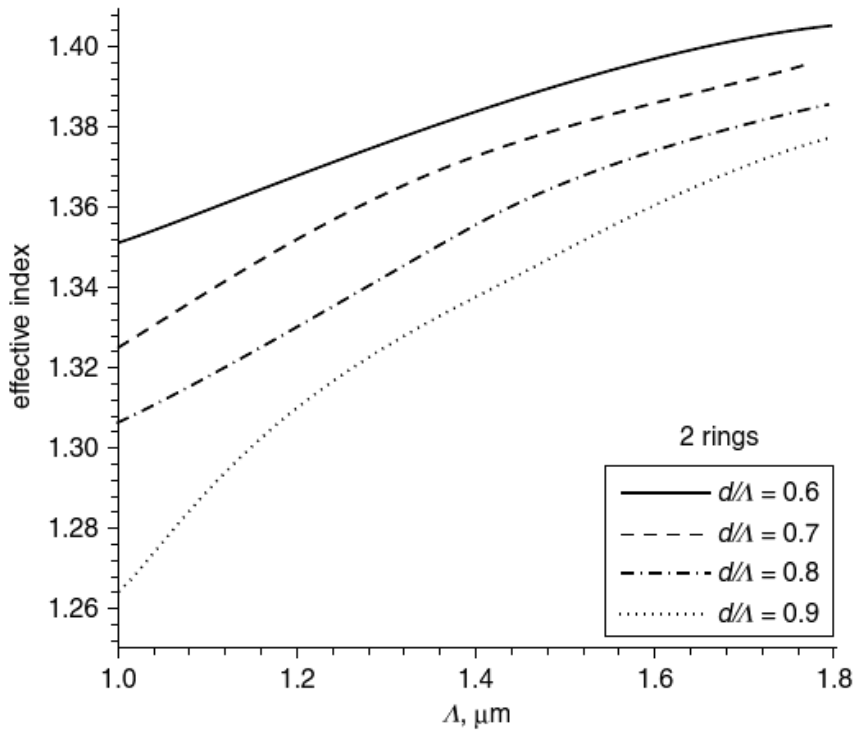
- Accuracy of coupling length as a function of ΔZ saturates below one wavelength

VBPM on a Photonic Crystal Fiber



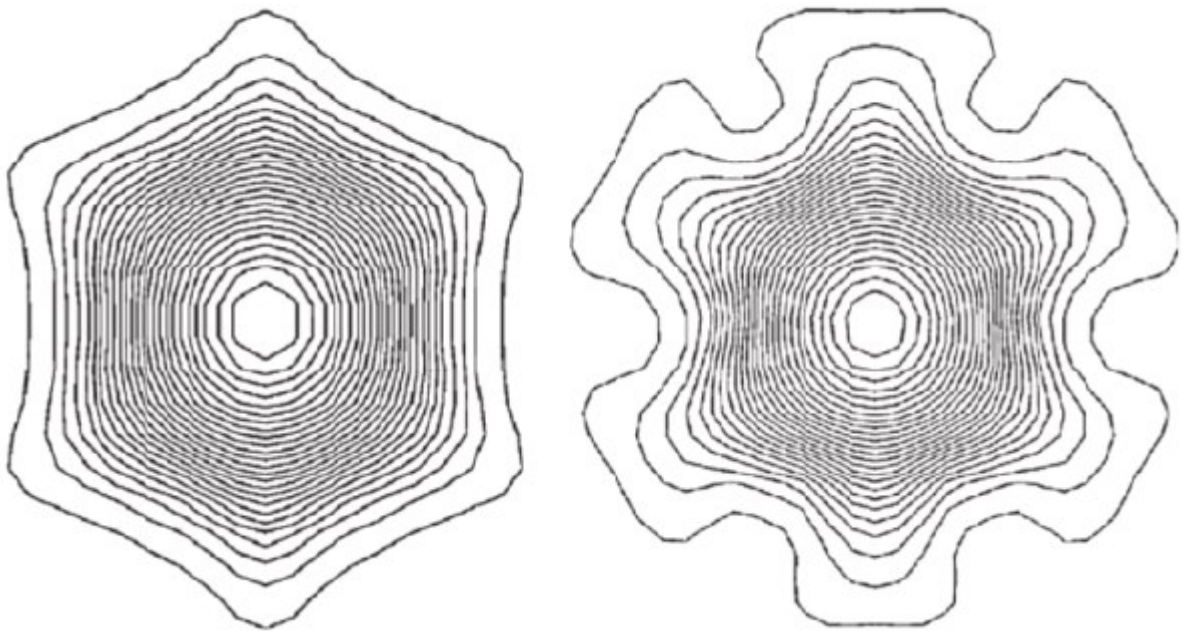
- Originally conceived of by P.J. Russell
- Confines light to core without total internal reflection!

VBPM on a PhC Fiber



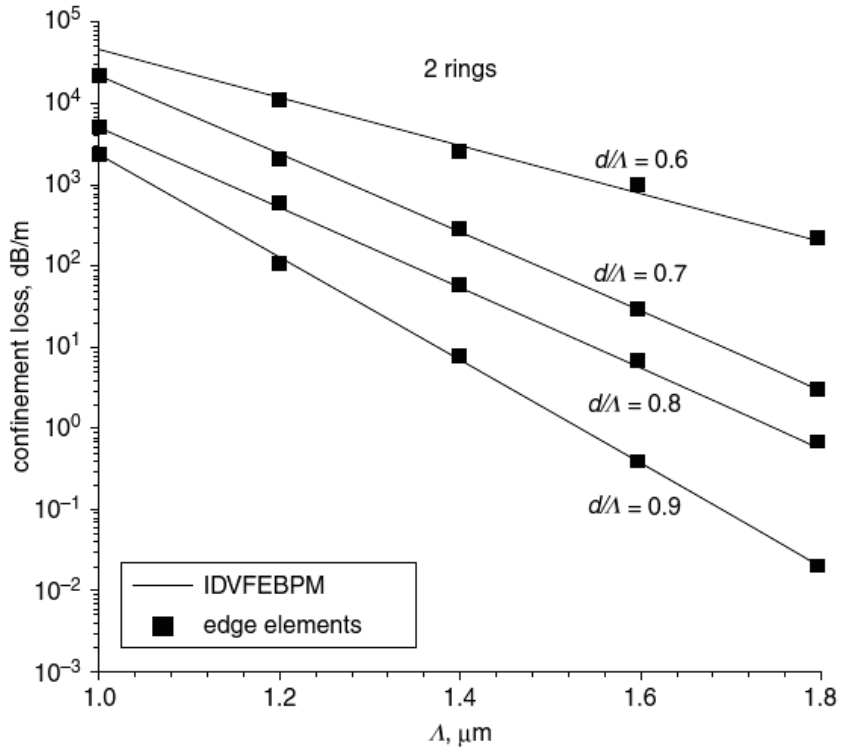
- Effective index vs. PhC period

VBPM on a PhC Fiber



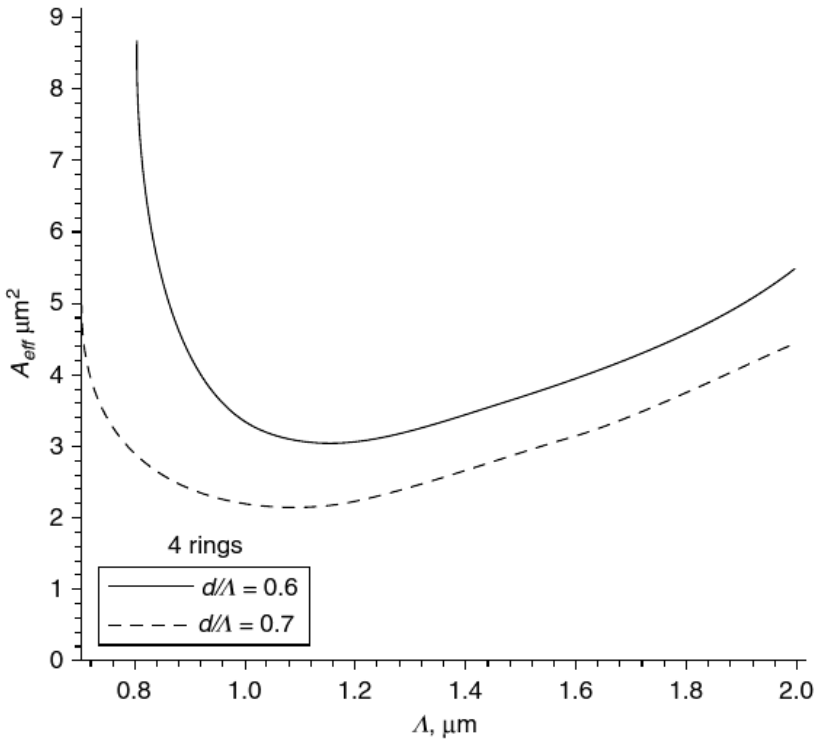
- H_y field distributions for the fundamental TE modes

VBPM on a PhC Fiber



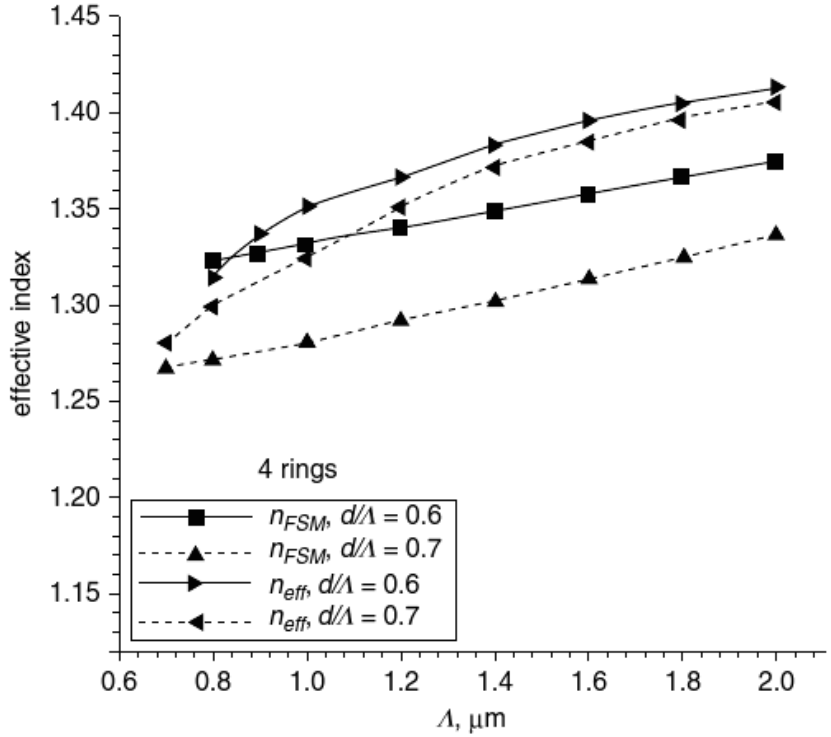
- Confinement loss decreases sharply as period Λ increases

VBPM on a PhC Fiber



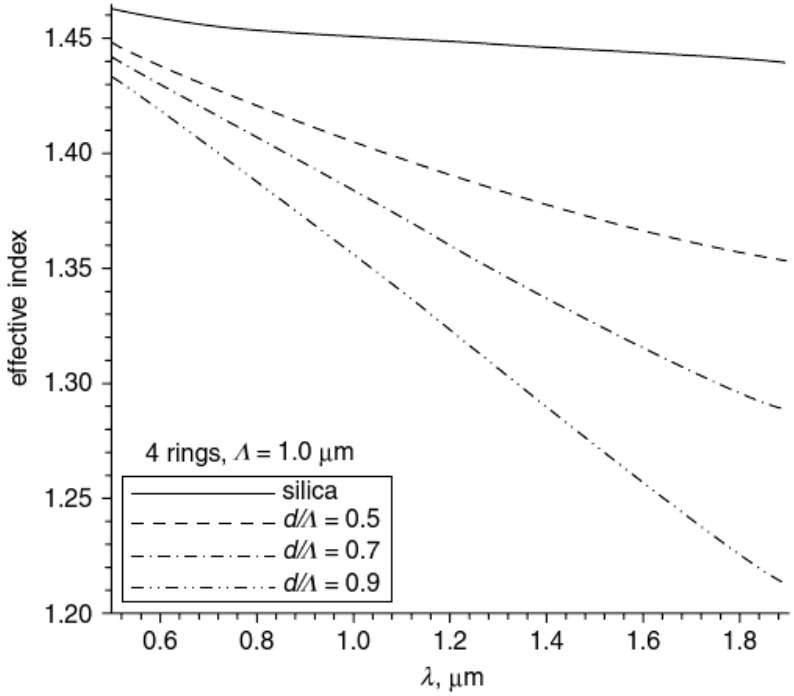
- Variation of the effective mode area with PhC period Λ

VBPM on a PhC Fiber



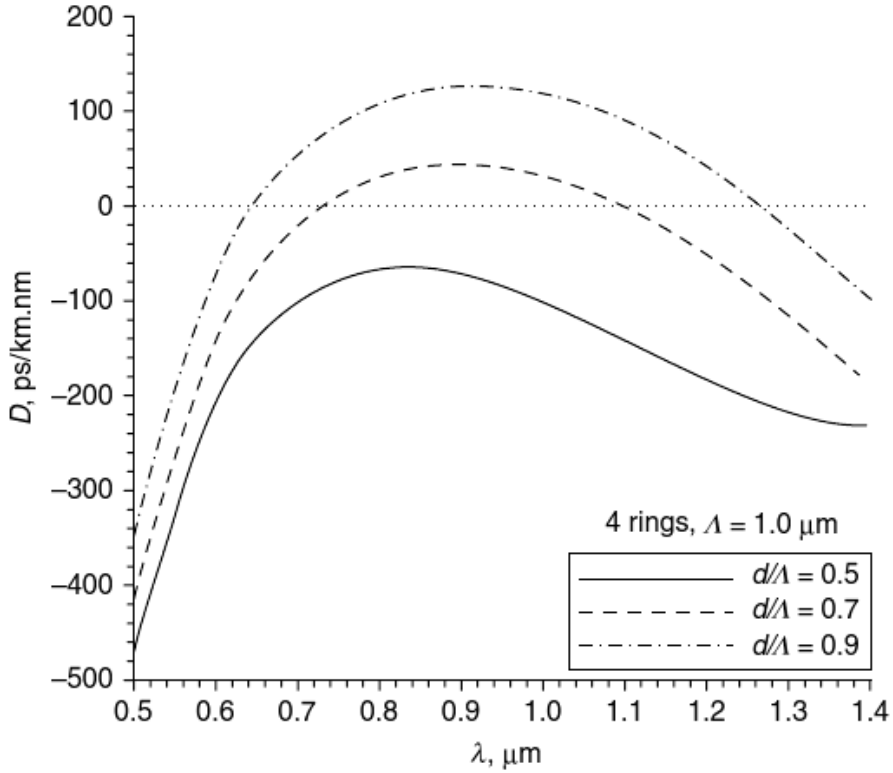
- Effective index increases modestly with increasing period Λ, indicating increased mode confinement

VBPM on a PhC Fiber



- Calculated dispersion relation (effective index versus wavelength) for a PhC Fiber

VBPM on a PhC Fiber



- Obtained dispersion $D = d^2k/d\omega^2$ from earlier data
- Note modest changes in parameters flip sign of *D*

Next Class

- Will discuss beam propagation method
- Recommended reading: Obayya, Chapter 2