To find \( \lim_{x \to c} f(x) \) analytically for a non-piecewise function \( f(x) \), we simply plug \( x = c \) into \( f(x) \) and see what happens.

There are three possible scenarios.

**Case 1:** \( f(c) \) returns a number.

It means that \( f(x) \) is continuous at \( x = c \).

In this case, \( \lim_{x \to c} f(x) = f(c) \), and we are done.

**Example 1:** Evaluate \( \lim_{x \to 4} (2x - 3) \) analytically.
Case 2: $f(c)$ returns $\frac{\text{nonzero number}}{0}$.

This means we have a vertical asymptote at $x = c$. The limit of $f(x)$ as $x$ approaches $c$ could be $\infty$, $-\infty$ or does not exist. We will need to analyze the left and the right-sided limits.

Example 2: Evaluate $\lim_{{x \to 1}} \frac{5}{(x-1)^2}$ analytically.

Example 3: Evaluate $\lim_{{x \to 1}} \frac{-5}{(x-1)^2}$ analytically.
Example 4: Evaluate \( \lim_{x \to 1} \frac{5}{x-1} \) analytically.

Case 3: \( f(c) \) returns \( \frac{0}{0} \).

This means \( f(x) \) has a hole or a vertical asymptote at \( x = c \) and so the limit of \( f(x) \) as \( x \) approaches \( c \) may or may not exist. What we need to do is to manipulate \( f(x) \) so that \( f(c) \) returns a number or \( \frac{\text{nonzero number}}{0} \) so that we can find the limits as in Case 1 or Case 2. To do this we factor the top and bottom and cancel.
Example 5: Evaluate $\lim_{x \to 3} \frac{x^3 - 3x^2}{x - 3}$ analytically.

Example 6: Evaluate $\lim_{x \to 2} \frac{x^2 - 2x}{(x - 2)^2}$ analytically.
Now we will talk about finding limits analytically for piecewise functions.

Example 7: Given: \( f(x) = \begin{cases} \frac{1}{x} & \text{if } x \leq 2 \\ -x & \text{if } x > 2 \end{cases} \)

a) Find \( \lim_{x \to 0} f(x) \)

b) Find \( \lim_{x \to 2} f(x) \)

c) Find \( \lim_{x \to 5} f(x) \)
### Finding Limits Analytically

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + 3$</td>
<td>$2x - 1$</td>
</tr>
</tbody>
</table>

- **$f(x)$**
  - $f(x) = x + 3$
  - $\lim_{x \to 7} f(x) = 10$

- **$g(x)$**
  - $g(x) = 2x - 1$
  - $\lim_{x \to 7} g(x) = 13$

<table>
<thead>
<tr>
<th>$f(x) \cdot g(x)$</th>
<th>$(x + 3)(2x - 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lim_{x \to 7} f(x) \cdot g(x)$</td>
<td>$52$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$f(x) + g(x)$</th>
<th>$(x + 3) + (2x - 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lim_{x \to 7} f(x) + g(x)$</td>
<td>$15$</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$f(x) - g(x)$</th>
<th>$(x + 3) - (2x - 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lim_{x \to 7} f(x) - g(x)$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$[f(x)]^2$</th>
<th>$(x + 3)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lim_{x \to 7} [f(x)]^2$</td>
<td>$52$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$f(x) / g(x)$</th>
<th>$\lim_{x \to 7} \left( \frac{f(x)}{g(x)} \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{7}{3}$</td>
</tr>
</tbody>
</table>
Properties of Limits:

Let $c$, $k$, $L$, and $K$ be real numbers and $n$ a positive integer.

If \( \lim_\limits{x \to c} f(x) = L \) and \( \lim_\limits{x \to c} g(x) = K \)

On the previous page:

$c = 7$, $f(x) = x + 3$, and $g(x) = 2x - 1$

\( \lim_\limits{x \to 7} f(x) = 7 + 3 = 10 \quad \lim_\limits{x \to 7} g(x) = 14 - 1 = 13 \)

\[
\begin{align*}
\lim_\limits{x \to c} [kf(x)] &= kL \\
\lim_\limits{x \to c} [f(x) \pm g(x)] &= L \pm K \\
\lim_\limits{x \to c} [f(x)g(x)] &= LK \\
\lim_\limits{x \to c} \left[ f(x) \right]^n &= L^n \\
\lim_\limits{x \to c} \left[ \frac{f(x)}{g(x)} \right] &= \frac{L}{K} \text{ assuming } K \neq 0
\end{align*}
\]

Example 8: Given:

\( \lim_\limits{x \to 2} f(x) = 5 \) and \( \lim_\limits{x \to 2} g(x) = 4 \)

Find: \( \lim_\limits{x \to 2} \left[ f^2(x) - 6g(x) - 3 \right] \)