We’ve seen how to solve quadratic equations \((ax^2 + bx + c = 0)\) by factoring and by extracting square roots, but what if neither of those methods are an option? What do we do with a quadratic equation that is not factorable and cannot be solved by extracting a square root? One option is to change a quadratic equation into a perfect square trinomial by using a procedure called **completing the square**.

The following table shows examples of perfect square trinomials in different forms. When a perfect square trinomial is in polynomial form, and the leading coefficient is 1, the constant term \(c\) is **ALWAYS** equal to half the coefficient of \(x\) squared.

<table>
<thead>
<tr>
<th>Perfect Squares</th>
<th>Factored Form</th>
<th>Polynomial Form</th>
<th>Constant Term = ((\frac{b}{2})^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x + 1)^2)</td>
<td>((x + 1)(x + 1))</td>
<td>(x^2 + 2x + 1)</td>
<td>(x^2 + 2x + \left(\frac{2}{2}\right)^2)</td>
</tr>
<tr>
<td>((x - 2)^2)</td>
<td>((x - 2)(x - 2))</td>
<td>(x^2 - 4x + 4)</td>
<td>(x^2 - 4x + \left(-\frac{4}{2}\right)^2)</td>
</tr>
<tr>
<td>((x - 3)^2)</td>
<td>((x - 3)(x - 3))</td>
<td>(x^2 - 6x + 9)</td>
<td>(x^2 - 6x + \left(-\frac{6}{2}\right)^2)</td>
</tr>
<tr>
<td>((x + 4)^2)</td>
<td>((x + 4)(x + 4))</td>
<td>(x^2 + 8x + 16)</td>
<td>(x^2 + 8x + \left(\frac{8}{2}\right)^2)</td>
</tr>
<tr>
<td>((x + 5)^2)</td>
<td>((x + 5)(x + 5))</td>
<td>(x^2 + 10x + 25)</td>
<td>(x^2 + 10x + \left(\frac{10}{2}\right)^2)</td>
</tr>
<tr>
<td>((x - 6)^2)</td>
<td>((x - 6)(x - 6))</td>
<td>(x^2 - 12x + 36)</td>
<td>(x^2 - 12x + \left(-\frac{12}{2}\right)^2)</td>
</tr>
</tbody>
</table>

This provides some insight into how we will be completing the square to get perfect squares. We must first be sure we are working with a quadratic equation with a leading coefficient of 1. Next, we will need to take the coefficient of \(x\), cut it in half, and then square it to get the constant term that is necessary to have a perfect square trinomial.
Completing the Square:
- Algorithm for solving quadratic equations \( ax^2 + bx + c = 0 \) that makes perfect squares which can be solved by extracting square roots
  - In order to use this algorithm, the leading coefficient of the quadratic equation must be 1

Steps for Completing the Square:
1. Divide each term by the leading coefficient \( a \)
   - If the leading coefficient is 1 \( (a = 1) \), skip this step
2. Isolate the constant term \( c \)
3. Add the square of half the coefficient of \( x \) to both sides \( \left(\frac{b}{2}\right)^2 \)
4. Factor the polynomial as a perfect square trinomial
5. Solve by extracting square roots

Example 1: Solve the equation \( x^2 - 6x - 5 = 0 \) for \( x \) and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter NO SOLUTION.

<table>
<thead>
<tr>
<th>( ac )</th>
<th>( b )</th>
<th>Think about the signs of the product and the sum.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 - 6x - 5 = 0 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The goal when solving an equation by completing the square is to take a polynomial equation that is not factorable and is not a perfect square, and make it a perfect square. The way we do that is by replacing the constant term of the original equation with a new constant term that not only allows us to factor, but that will always result in a polynomial that factors as a perfect square. The new constant term will **ALWAYS** be the coefficient of $x$ divided by 2 squared $\left(\left(\frac{b}{2}\right)^2\right)$.

Keep in mind that not every polynomial equation has real solutions, regardless of how you solve the equation (extracting roots, completing the square, quadratic formula, etc.).

**Example 2:** Solve the equation $-3x^2 + 6x - 48 = 0$ for $x$ and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter **NO SOLUTION**.

<table>
<thead>
<tr>
<th>$ac$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>$-2$</td>
</tr>
</tbody>
</table>

Since there are not two numbers whose product is 16 and whose sum is $-2$, the trinomial $x^2 - 2x + 16 = 0$ is not factorable.

$$-3x^2 + 6x - 48 = 0$$

$$\frac{-3x^2}{-3} + \frac{6x}{-3} - \frac{48}{-3} = \frac{0}{-3}$$

$$x^2 - 2x + 16 = 0$$

$$x^2 - 2x = -16$$

$$x^2 - 2x + \left(\frac{-2}{2}\right)^2 = -16 + \left(\frac{-2}{2}\right)^2$$

$$x^2 - 2x + 1 = -16 + 1$$

$$(x - 1)(x - 1) = -15$$

$$(x - 1)^2 = -15$$
No real number squared (a real number times itself) can equal a negative value. Therefore when we see the equation \((x - 1)^2 = -15\), this tells us there will be no real solution for this quadratic equation. However if you don’t recognize that, you could still go on to the next step and take the square root of both sides of the equation.

\[
(x - 1)^2 = -15
\]

\[
x - 1 = \pm \sqrt{-15}
\]

You cannot take the square root of a negative number and get a real number in return. Therefore there is NO SOLUTION for the quadratic equation \(-3x^2 + 6x - 48 = 0\). This would be the case regardless of how we chose to solve the equation.

**Example 3:** Solve the following equations for \(x\) and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter NO SOLUTION.

a. \(-x^2 - 9x + 3 = 0\)  

b. \(2x^2 + 8x = -4\)
c. $x^2 + 4x + 13 = 0$

d. $3x^2 = 6x - 1$

\[
\frac{3x^2}{3} = \frac{6x}{3} - \frac{1}{3}
\]

\[
x^2 = 2x - \frac{1}{3}
\]

\[
x^2 - 2x = -\frac{1}{3}
\]

\[
x^2 - 2x + \left(\frac{-2}{2}\right)^2 = -\frac{1}{3} + \left(\frac{-2}{2}\right)^2
\]

\[
x^2 - 2x + 1 = -\frac{1}{3} + 1
\]

\[
(x - 1)^2 = \frac{2}{3}
\]

\[
x - 1 = \pm \sqrt{\frac{2}{3}}
\]

\[
x = 1 \pm \sqrt{\frac{2}{3}}
\]
Lesson 13

Solving Quadratic Equations by Completing the Square

**e.** \[3x^2 + 12x - 39 = 0\]

\[
\begin{align*}
\frac{3x^2}{3} + \frac{12x}{3} - \frac{39}{3} &= 0 \\
x^2 + 4x - 13 &= 0 \\
x^2 + 4x &= 13 \\
x^2 + 4x + \left(\frac{4}{2}\right)^2 &= 13 + \left(\frac{4}{2}\right)^2 \\
x^2 + 4x + 4 &= 13 + 4 \\
(x + 2)(x + 2) &= 17 \\
(x + 2)^2 &= 17 \\
x + 2 &= \pm\sqrt{17} \\
x &= -2 \pm \sqrt{17} \\
x &= -2 + \sqrt{17}, -2 - \sqrt{17}
\end{align*}
\]

\[f.\ 5x^2 + x + 2 = 0\]

\[
\begin{align*}
x^2 + \frac{1}{5}x - \frac{2}{5} &= 0 \\
x^2 + \frac{1}{5}x &= \frac{2}{5} \\
x^2 + \frac{1}{5}x + \left(\frac{1}{10}\right)^2 &= \left(\frac{1}{10}\right)^2 + \frac{2}{5} \\
x^2 + \frac{1}{5}x + \frac{1}{100} &= \frac{1}{100} + \frac{2}{5} \\
100x^2 + 20x + 1 &= 1 + 40 \\
100x^2 + 10x + 10x + 1 &= 41 \\
(x + 2)(x + 2) &= 17 \\
(x + 2)^2 &= 17 \\
10(10x + 1) + 1(10x + 1) &= 41 \\
(x + 1)(10x + 1) &= 41 \\
(10x + 1)^2 &= 41 \\
10x + 1 &= \pm\sqrt{41} \\
10x &= -1 \pm \sqrt{41} \\
x &= \frac{-1 \pm \sqrt{41}}{10}
\end{align*}
\]

Notice in the solution for the quadratic equation in part **f.** that I eliminated the fractions by multiplying each term by 100. I was still able to factor using the ac-method (and then by grouping), and I still ended up with a perfect square. Feel free to use this method if you’d like, or disregard it.
Finally, just like with factoring, completing the square is a method of solving equations that will be used for more than just solving quadratic equations. Other polynomial equations such as $x^4 - 3x^2 + 1 = 0$ (which we will see in future lessons) are not quadratic but can still be solved by completing the square. Completing the square can also be used when working with quadratic functions.

**Answers to Exercises:**

1. $x = 3 - \sqrt{14}, 3 + \sqrt{14}$; 2. *NO SOLUTION*;
3a. $x = -\frac{9}{2} - \frac{\sqrt{93}}{2}, -\frac{9}{2} + \frac{\sqrt{93}}{2}$; 3b. $x = -2 - \sqrt{2}, -2 + \sqrt{2}$;
3c. *NO SOLUTION*; 3d. $x = 1 - \sqrt{\frac{2}{3}}, 1 + \sqrt{\frac{2}{3}}$;
3e. $x = -2 - \sqrt{17}, -2 + \sqrt{17}$; 3f. $x = \frac{-1 + \sqrt{41}}{10}, \frac{-1 - \sqrt{41}}{10}$;