In the previous lesson we showed how to solve quadratic equations that were not factorable and were not perfect squares by making perfect square trinomials using a process called completing the square. In this lesson we will see another method for solving quadratic equations which are not factorable and are not perfect squares by using a formula called **the quadratic formula**, which is derived from completing the square.

**Quadratic Formula:**
- another method for solving quadratic equations \((ax^2 + bx + c = 0)\)
  \[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
- the result of completing the square for the standard quadratic equation \(ax^2 + bx + c = 0\)
  - if you took \(ax^2 + bx + c = 0\) and followed the steps for completing the square, you would derive the quadratic formula
- unlike when solving quadratic equations by completing the square, you do **NOT** have to make the leading coefficient \(a = 1\), however since we are working with equations we can multiply and/or divide both sides of the equation by anything we’d like in order to make it easier to work with
  - \(-0.6x^2 + 1.3x - 1 = 0\)
  - \(\frac{2}{3}x^2 - \frac{1}{6}x - 4 = 0\)
Example 1: Solve the following equations for $x$ and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter NO SOLUTION.

\[ x^2 + 3x + 1 = 0 \]
Steps for Solving Quadratic Equations using the Quadratic Formula:

1. write the equation in polynomial form and set it equal to zero
   \[ ax^2 + bx + c = 0 \]

2. identify the coefficients \( a, b, \) and \( c, \) and plug them into the formula
   \[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

3. **SIMPPLY COMPLETELY**
   - simplify your square root completely
     - look for perfect squares \((4, 9, 16, \ldots)\), or for numbers that contain factors that are perfect squares \((8, 45, 96, \ldots)\)
   - simplify the fraction completely
     - factor the numerator and cancel common factors with the denominator, or write two separate fractions and simplify each

Plugging the quadratic equation \(7x^2 - 2x - 7 = 0\) into the quadratic formula results in the following:

\[
x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(7)(-7)}}{2(7)}
\]

\[
x = \frac{2 \pm \sqrt{4 + 196}}{14}
\]

\[
x = \frac{2 \pm \sqrt{200}}{14}
\]

\[
x = \frac{2 \pm 10\sqrt{2}}{14}
\]

\[
x = \frac{2}{14} \pm \frac{10\sqrt{2}}{14}
\]

\[
x = \frac{1}{7} \pm \frac{5\sqrt{2}}{7}
\]

\[
x = \frac{1}{7} + \frac{5\sqrt{2}}{7}, \quad \frac{1}{7} - \frac{5\sqrt{2}}{7}
\]

200 is not a perfect square, but it contains a factor of 100 which is, so that is why I broke \(\sqrt{200}\) into \(\sqrt{100}\sqrt{2}\).

To simplify the fraction, you could factor out a 2 from the numerator and then cancel with the denominator, or you can break the fraction \(\frac{2 \pm 10\sqrt{2}}{14}\) into two separate fractions, \(\frac{2}{14}\) and \(\frac{10\sqrt{2}}{14}\) in order to simplify each individually. Either way is acceptable.
**Example 2:** Solve the following equations for \( x \) and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter NO SOLUTION.

a. \( x^2 + 10x + 12 = 0 \)  
   
   b. \( 3x^2 - 3x - 4 = 0 \)  

\[
\begin{align*}
3x^2 - 2x + 1 &= 0 \\
\Rightarrow x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(1)}}{2(3)} \\
&= \frac{2 \pm \sqrt{4 - 12}}{6} \\
&= \frac{2 \pm \sqrt{-8}}{6} \\
\end{align*}
\]

Since the square root of \(-8\) (or negative anything) does not exist with real numbers, the equation \( 3x^2 = 2x - 1 \) has **NO SOLUTION**.
**Example 3:** Solve the following equations for $x$ and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter NO SOLUTION.

a. \[ \frac{2x}{x-3} + \frac{5}{x+3} = \frac{36}{x^2-9} \]

b. \[ 1 = \frac{5}{x} - \frac{20}{x^2} \]
c. \( \frac{1}{x} + \frac{1}{x+2} = \frac{1}{4} \)

d. \( \frac{3}{x-3} + \frac{5}{x-4} = \frac{x^2-19}{x^2-7x+12} \)

Multiply both sides of the equation by \( x, x + 2, \) and 4 to eliminate the fractions.

\[
4x(x + 2) \left( \frac{1}{x} + \frac{1}{x+2} \right) = \left( \frac{1}{4} \right) 4x(x + 2)
\]

\[
4(x + 2) + 4x = x(x + 2)
\]

\[
4x + 8 + 4x = x^2 + 2x
\]

\[
8x + 8 = x^2 + 2x
\]

\[
0 = x^2 - 6x - 8
\]

\[
x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-8)}}{2(1)}
\]

\[
x = \frac{6 \pm \sqrt{36 + 32}}{2}
\]

\[
x = \frac{6 \pm \sqrt{68}}{2}
\]

\[
x = \frac{6 \pm 2\sqrt{17}}{2}
\]

\[
x = \frac{6}{2} \pm \frac{2\sqrt{17}}{2}, \quad \frac{6}{2} - \frac{2\sqrt{17}}{2}
\]

\[
x = 3 + \sqrt{17}, \ 3 - \sqrt{17}
\]
Multiply both sides of the equation by \((x + 4)\) and \((x + 2)\) to eliminate the fractions.

\[-2x = x(x + 2) - 2(x + 4)\]
\[-2x = x^2 + 2x - 2x - 8\]
\[0 = x^2 + 2x - 8\]

\[x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-8)}}{2(1)}\]
\[x = \frac{-2 \pm \sqrt{4 + 32}}{2}\]
\[x = \frac{-2 \pm 6}{2}\]
\[x = \frac{-2 + 6}{2} , \frac{-2 - 6}{2}\]
\[x = 2 , -4\]

\(x = -4\) is not a valid answer because it would make the original equation have denominators of zero. So the only valid answer is \(x = 2\).
Every quadratic equation can be solved by either completing the square or by using the quadratic formula. While many students prefer the quadratic formula, keep in mind that the quadratic formula is limited to solving only quadratic equations, while completing the square can be used to solve other non-quadratic equations (as we will see in Lesson 15).

Answers to Exercises:

1. \( x = \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2} \); 2a. \( x = -5 - \sqrt{13}, -5 + \sqrt{13} \);

2b. \( x = \frac{3-\sqrt{57}}{6}, \frac{3+\sqrt{57}}{6} \); 2c. NO SOLUTION;

2d. \( x = -5 - 4\sqrt{2}, -5 + 4\sqrt{2} \); 3a. \( x = -\frac{17}{2} \); 3b. NO SOLUTION;

3c. \( x = 3 + \sqrt{17}, 3 - \sqrt{17} \); 3d. \( x = 4 - 2\sqrt{2}, 4 + 2\sqrt{2} \);

3e. \( x = -5, -1 \); 3f. \( x = 2 \);