We have seen already when covering Lesson 3 that fractional exponents are simply an alternate way of expressing radicals.

$$\sqrt{6} = (6)^{\frac{1}{2}}$$

So a square root is equivalent to a power of $\frac{1}{2}$, which is the reciprocal of the index 2. The same is true for any radical; to express a radical as an exponent, we simply need to take the reciprocal of the index of the radical.

$$\sqrt[3]{5} = (5)^{\frac{1}{3}}$$
$$\sqrt[4]{4} = (4)^{\frac{1}{4}}$$
$$\sqrt[5]{3} = (3)^{\frac{1}{5}}$$
$$\sqrt[6]{2} = (2)^{\frac{1}{6}}$$

This led us to our notation for expressing radicals using fractional exponents, as well as converting fractional exponents back to radicals, which we will be focusing on in this lesson.

<u>Converting an exponent $\left(x^{\frac{1}{n}}\right)$ to a radical $\left(\sqrt[n]{x}\right)$</u>

- to write a fractional exponent as a radical, write the denominator of the exponent as the index of the radical and the base of the expression as the radicand
  - the expression $x^{\frac{3}{5}}$ can be written as a radical in two ways, both of which are equivalent
    - $x^{\frac{3}{5}} = \sqrt[5]{x^3}$
    - $x^{\frac{3}{5}} = \left(\sqrt[5]{x}\right)^3$
  - regardless of which way you choose to write the radical expression, the index is the same (in this case 5)
- it is imperative that the radical exists, otherwise the expression is meaningless; in other words, if the radical has an **<u>even root</u>**, the radicand must be **<u>non-negative</u>**

Steps for Solving Equations with Fractional Exponents:

1. isolate the variable that has a fractional exponent
2. convert from a fractional exponent to a radical
3. solve for the variable by using roots and/or exponents (principle of powers)

Example 1: Solve the following equations for x and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter NO SOLUTION.

a. $x^{\frac{3}{5}} = 27$

b. $2x^{\frac{5}{4}} + 10 = 74$

$$2x^{\frac{5}{4}} = 64$$

$$x^{\frac{5}{4}} = 32$$

$$\left(\sqrt[4]{x}\right)^5 = 32$$

To eliminate the exponent of 5 that's outside the parentheses, take the fifth root of both sides of the equation:

$$\sqrt[5]{\left(\sqrt[4]{x}\right)^5} = \sqrt[5]{32}$$

Keep in mind that taking the odd root of both sides of an equation does not produce both a positive and negative root; that only happens with an even root:

$$\sqrt[4]{x} = 2$$

Now to eliminate the fourth root, take both sides of the equation to the power of 4:

$$\left(\sqrt[4]{x}\right)^4 = (2)^4$$

$$\boldsymbol{x = 16}$$

<u>Example 2:</u> Solve the following equations for x and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter NO SOLUTION.

a. $x^2 = 16$

$x = \pm\sqrt{16}$

$\boldsymbol{x = \pm 4}$

When taking the square root of both sides of an equation, there is a positive root and a negative root. This is because solving an equation such as $x^2 = 16$ by extracting roots must produce the same answer as if we had solved by factoring.

$x^2 = 16$

$x^2 - 16 = 0$

$(x-4)(x+4) = 0$

$x - 4 = 0 \quad ; \quad x + 4 = 0$

$\boldsymbol{x = 4 \quad ; \quad x = -4}$

b. $x^{\frac{2}{3}} = 16$

Example 3: Solve the following equations for x and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter NO SOLUTION.

a. $x^2 = -25$

b. $x^{\frac{2}{3}} = -25$

Using real numbers, it is not possible to have something squared equal to a negative value. So in this case, there is no real number that you can raise to the power of 2, and produce -25. Therefore $x^2 = -25$ has no real solutions.

NO SOLUTION

If you did not realize that $x^2 = -25$ did not have any real solutions, you could still attempt to take the square root of both sides of the equation in order to solve for x.

$x^2 = -25$

$x = \pm\sqrt{-25}$

Since it is not possible to take the square root of a negative value using real numbers, now you should see there is no real solution for this equation.

NO SOLUTION

<u>Example 4:</u> Solve the following equations for x and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter NO SOLUTION.

a. $\sqrt{x} = -64$

b. $x^{\frac{3}{2}} = -64$

Using real numbers, it is not possible to have the square root of something equal to a negative value. So in this case, there is no real number that you can take the square root of to produce -64. Therefore $\sqrt{x} = -64$ has no real solutions.

NO SOLUTION

If you did not realize that $\sqrt{x} = -64$ has no real solutions, you could still attempt to solve the problem by squaring both sides of the equation.

$\sqrt{x} = -64$

$x = 4096$

However the answer of 4096 does not make the original equation true, because the square root of 4096 is 64, not -64.

NO SOLUTION

The three previous examples all show special cases of equations with fractional exponents that could result in more than one real solution (Example 2), or in no real solutions (Examples 3 & 4). Notice that all three special cases contain either an even number in the numerator of the fractional exponent, or an even number in the denominator of the fractional exponent. Keep in mind that when you take the even root of both sides of an equation, you get a positive root and a negative root (Example 2). Also keep in mind that anything taken to an even power must be non-negative (Example 4), and that you cannot take the even root of a negative number (Example 3).

Since equations with fractional exponents that have no solution (like Examples 3 and 4 above) tend to reveal themselves during the process of solving, it is not imperative that we check our answers like we do with radical equations (even though I usually write my fractional equations using radicals). However if you are not catching things like square roots being equal to a negative value, or a squared value being equal to a negative value while solving, you may need to check your answers by plugging them back in to the original equation to see whether you get a true statement or not.

<u>Example 5:</u> Solve the following equations for x and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter NO SOLUTION.

a. $(2x-3)^{\frac{9}{7}} = -1$

b. $\left(\frac{x}{4}\right)^{\frac{2}{3}} - 5 = 20$

c. $-8x^{-\frac{3}{2}} = 1$

d. $x^{\frac{2}{3}} - 5 = 20$

e. $x^{\frac{2}{3}} = -\frac{1}{9}$

$\left(\sqrt[3]{x}\right)^2 = -\frac{1}{9}$

It is not possible to take any real number, square it, and get a negative number in return. So using real numbers, $\left(\sqrt[3]{x}\right)^2 = -\frac{1}{9}$ is not possible. Therefore the correct answer for this is **NO SOLUTION**.

f. $x^{-\frac{3}{5}} = 0.001$

$\frac{1}{x^{\frac{3}{5}}} = \frac{1}{1000}$

$1000 = x^{\frac{3}{5}}$

$1000 = \left(\sqrt[5]{x}\right)^3$

To isolate x I'll take the cubed root of both sides to get rid of the power of 3 then I'll raise both sides to is the power of 5.

$1000 = \left(\sqrt[5]{x}\right)^3$

$\sqrt[3]{1000} = \sqrt[5]{x}$

$10 = \sqrt[5]{x}$

$(10)^5 = x$

$x = 100{,}000$

Answers to Examples:

1a. $x = 243$; *1b.* $x = 16$; *2a.* $x = -4, 4$; *2b.* $x = -64, 64$;
3a. NO SOLUTION ; *3b.* NO SOLUTION ; *4a.* NO SOLUTION ;
4b. NO SOLUTION ; *5a.* $x = 1$; *5b.* $x = -500, 500$;
5c. NO SOLUTION ; *5d.* $x = -125, 125$; *5e.* NO SOLUTION ;
5f. $x = 100{,}000$;