A difference quotient is an expression that represents the difference between two function values divided by the difference between two inputs. This is an extension of the slope formula from Lessons 16 and 17 $\left(m = \frac{\Delta y}{\Delta x}\right)$, when we found the change in y (or the difference between two y values) and divided by the change in x. Now we will find the difference between two function values, divided by the difference between two inputs:

$$\frac{\Delta f(x)}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x}$$

By combining like terms in the denominator, we get the following simplified form of a difference quotient:

$$\frac{\Delta f(x)}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

<u>Difference Quotient</u>:

- a fraction (or quotient) containing the difference of two functions values in the numerator, and the difference of two inputs in the denominator
  - $\frac{f(x+h)-f(x)}{h}$
  - the input x could be replaced with a numeric value or another expression

Our focus when working with difference quotients in this class is to simplify them. To do so, I prefer to follow the step-by-step procedure which is demonstrated on the next page, but you are welcome to use another method if you choose.

Example 1: Given $f(x) = 5x - 2$, find the difference quotient $\frac{f(a+h)-f(a)}{h}$.

Steps for Simplifying a Difference Quotient:

1. find the first function value
   - in Example 1 I find $f(a+h)$ by replacing x in the function $f(x) = 5x - 2$ with the expression $a + h$

$$f(a+h) = 5(a+h) - 2$$

$$5a + 5h - 2$$

2. find the second function value
   - I find $f(a)$ by replacing x in the function $f(x) = 5x - 2$ with the expression a

$$f(a) = 5a - 2$$

3. find the difference between the two function values
   - I find $f(a+h) - f(a)$ by taking the two function values from steps 1 and 2 and subtracting them

$$f(a+h) - f(a) = 5a + 5h - 2 - (5a - 2)$$

$$5a + 5h - 2 - 5a + 2$$

$$5h$$

4. divide the difference by the expression h
   - since a difference quotient is a fraction, be sure to simplify completely by factoring and canceling common factors

$$\frac{f(a+h)-f(a)}{h} = \frac{5h}{h}$$

$$\mathbf{5}$$

Example 2: Given $j(x) = -x^2 - x + 7$, find the difference quotient $\frac{j(3+h)-j(3)}{h}$.

a. $j(3+h) = -(3+h)^2 - (3+h) + 7$

$= -(3+h)(3+h) - 3 - h + 7$

$= -(9 + 6h + h^2) - 3 - h + 7$

$= -9 - 6h - h^2 - 3 - h + 7$

$\boldsymbol{= -h^2 - 7h - 5}$

b. $j(3) = -3^2 - 3 + 7$

$= -9 - 3 + 7$

$\boldsymbol{= -5}$

Notice that after replacing x with $3 + h$ in the function j, we basically do addition, subtraction, and multiplication with polynomials, just like we've already done in Lesson 5.

Be aware on part b. that -3^2 is the same as $-1 \cdot 3^2$, which is why it simplifies to -9.

c. $j(3+h) - j(3) =$

d. $\frac{j(3+h)-j(3)}{h} =$

Example 3: Given the function $f(x) = 2x - 3$, find the difference quotient $\frac{f(a+h)-f(a)}{h}$.

Steps for Determining the Value of a Difference Quotient:

1. find the first function value

2. find the second function value

3. find the difference between the two function values

4. divide the difference by the expression h

Again, I prefer to break difference quotients into smaller pieces in order to simplify them, but you do not have to. You can go through and simplify difference quotients by leaving them as one single expression the entire time, as demonstrated in the next example.

<u>Example 4:</u> Given the function $m(x) = -5x^2 + 10x$, find the difference quotient $\frac{m(x+h)-m(x)}{h}$.

$$\frac{m(x+h)-m(x)}{h} = \frac{-5(x+h)^2 + 10(x+h) - (-5x^2 + 10x)}{h}$$

$$\frac{-5(x+h)(x+h) + 10x + 10h + 5x^2 - 10x}{h}$$

$$\frac{-5(x^2 + 2xh + h^2) + 10x + 10h + 5x^2 - 10x}{h}$$

$$\frac{-5x^2 - 10xh - 5h^2 + 10x + 10h + 5x^2 - 10x}{h}$$

$$\frac{-10xh - 5h^2 + 10h}{h}$$

$$\boldsymbol{-10x - 5h + 10}$$

When simplifying difference quotients, use whichever procedure makes the most sense to you.

Example 5: Given the function $j(x) = \frac{1}{3}x^2 + 5x$, find the difference quotient $\frac{j(-2+h)-j(-2)}{h}$.

$$j(-2+h) = \frac{1}{3}(-2+h)^2 + 5(-2+h)$$

$$j(-2+h) = \frac{1}{3}(-2+h)(-2+h) - 10 + 5h$$

$$j(-2+h) = \frac{1}{3}(4 - 4h + h^2) - 10 + 5h$$

$$j(-2+h) = \frac{4}{3} - \frac{4}{3}h + \frac{1}{3}h^2 - 10 + 5h$$

$$j(-2+h) = \frac{1}{3}h^2 + \frac{11}{3}h - \frac{26}{3}$$

$$j(-2) = \frac{1}{3}(-2)^2 + 5(-2)$$

$$j(-2) = \frac{1}{3}(4) - 10$$

$$j(-2) = \frac{4}{3} - 10$$

$$j(-2) = -\frac{26}{3}$$

$$j(-2+h) - j(-2) = \frac{1}{3}h^2 + \frac{11}{3}h - \frac{26}{3} - \left(-\frac{26}{3}\right)$$

$$j(-2+h) - j(-2) = \frac{1}{3}h^2 + \frac{11}{3}h - \frac{26}{3} + \frac{26}{3}$$

$$j(-2+h) - j(-2) = \frac{1}{3}h^2 + \frac{11}{3}h$$

$$\frac{j(-2+h)-j(-2)}{h} = \frac{\frac{1}{3}h^2 + \frac{11}{3}h}{h}$$

$$\frac{j(-2+h)-j(-2)}{h} = \frac{\frac{1}{3}h^2}{h} + \frac{\frac{11}{3}h}{h}$$

$$\boldsymbol{\frac{j(-2+h)-j(-2)}{h} = \frac{1}{3}h + \frac{11}{3}}$$

As stated before, there is a lot of review from Lesson 5 when simplifying a difference quotient. As shown in Example 5 on the left, we have to multiply binomials and use the distributive property to find the function value $j(-2+h)$.

We also have to combine like terms to simplify that function value as much as possible by adding and subtracting terms.

And finally, when simplifying the quotient, I broke the two terms in the numerator into two separate fractions to simplify completely. These are all concepts that were covered in Lesson 5; feel free to review the Lesson 5 notes, if necessary.

<u>Example 6:</u> Given the function $m(x) = (1 - x)^2$, find the difference quotient $\frac{m(a+h)-m(a)}{h}$.

Example 7: Given the function $n(x) = \frac{1}{x}$, find the difference quotient $\frac{n(x+h)-n(x)}{h}$.

$$\frac{n(x+h)-n(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$\frac{\frac{x}{x} \cdot \frac{1}{x+h} - \frac{1}{x} \cdot \frac{x+h}{x+h}}{h}$$

$$\frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h}$$

$$\frac{\frac{x-(x+h)}{x(x+h)}}{h}$$

$$\frac{x-x-h}{x(x+h)} \div h$$

$$\frac{-h}{x(x+h)} \cdot \frac{1}{h}$$

$$\frac{-h}{xh(x+h)}$$

$$\boldsymbol{\frac{-1}{x(x+h)}}$$

Answers to Examples:

1. $\frac{f(a+h)-f(a)}{h} = 5$; *2a.* $j(3+h) = -h^2 - 7h - 5$; *2b.* $j(3+h) = -5$;

2c. $j(3+h) - j(3) = -h^2 - 7h$; *2d.* $\frac{j(3+h)-j(3)}{h} = -h - 7$;

3. $\frac{f(a+h)-f(a)}{h} = 2$; *4.* $\frac{m(x+h)-m(x)}{h} = -10x - 5h + 10$; *5.* $\frac{j(-2+h)-j(-2)}{h} = \frac{1}{3}h + \frac{11}{3}$;

6. $\frac{m(a+h)-m(a)}{h} = -2 + h + 2a$; *7.* $\frac{n(x+h)-n(x)}{h} = \frac{-1}{x(x+h)}$;