In this lesson we will be covering linear functions. Much of the material covered in this lesson will be a review of linear equations. However it is important to be aware of the notation that we will use; $y = mx + b$ is the notation for a linear equation while $f(x) = mx + b$ is the notation for a linear function.

<u>Linear functions:</u>

- functions defined by the linear expression $mx + b$
  - $f(x) = mx + b$
    - m is the slope $\left(m = \frac{\Delta f(x)}{\Delta x} = \frac{\Delta \text{output}}{\Delta \text{input}} = \frac{\updownarrow \Delta}{\leftrightarrow \Delta}\right)$
    - $(0, b)$ is point where the graph crosses the vertical axis
- to find the expression that defines a linear function, we can still use point-slope form $y - y_1 = m(x - x_1)$ by replacing y with $f(x)$
  - $f(x) - f(x_1) = m(x - x_1)$, where m is the slope and $(x_1, f(x_1))$ is an ordered pair that the graph of the line passes through
  - we could also use slope-intercept form, again replacing y with $f(x)$
    - $f(x) = mx + b$
- just like when we found linear equations $(y = mx + b)$, we need one point that a line passes through and the slope of the line to find a linear function $(f(x) = mx + b)$
  - if you are not given the slope of a line, you can find it using any two points that the line passes through

As stated in the previous lesson, the inputs of a function (the elements of the domain) are always the values on the horizontal axis (x-axis) and the outputs (the elements of the range) are always the values on the vertical axis (y-axis). This means ordered pairs are written $(\text{input}, \text{output})$. So for a linear function $f(x) = mx + b$, the ordered pairs would be written $(\text{input}, \text{output})$ or $(x, f(x))$. That means a function value such as $f(-3) = 4$ would be written as the ordered pair $(-3, 4)$.

Example 1: Given the function values $f(1) = -4$ and $f(5) = 8$, complete the following:

a. Write the function values as ordered pairs.

The ordered pairs of a function are written as $(\text{input}, \text{output})$ or $(x, f(x))$. The function value $f(1) = -4$ indicates that when 1 is the input, -4 is the output, so the ordered pair is $(1, -4)$. The function value $f(5) = 8$ indicates that when $x = 5$, the function value is 8, so the ordered pair is $(5, 8)$.

b. Use the ordered pairs from part a. to find the slope of the line that passes through those two points.

When working with linear equations, we expressed the slope formula as $\frac{\Delta y}{\Delta x}$. Since we are now working with linear functions, I will replace y with $f(x)$ to get $m = \frac{\Delta f(x)}{\Delta x}$. Using the ordered pairs $(1, -4)$ and $(5,8)$ from part a., I get the following for the slope:

$$m = \frac{\Delta f(x)}{\Delta x} = \frac{12}{4} = 3$$

c. Write a linear function $f(x) = mx + b$ using the slope from part b. and one of the ordered pairs from part a.

Now that we have the slope of the linear function, as well as two ordered pairs, we can find the function using point-slope form. I will use the point $(5,8)$ and the slope of 3.

$$f(x) - f(x_1) = m(x - x_1)$$

$$f(x) - 8 = 3(x - 5)$$

$$f(x) - 8 = 3x - 15$$

$$\boldsymbol{f(x) = 3x - 7}$$

Sometimes using new notation such as $(x, f(x))$ or $(\text{input}, \text{output})$ for ordered pairs, or $m = \frac{\Delta f(x)}{\Delta x}$ for the slope, or $f(x) - f(x_1) = m(x - x_1)$ for point slope form can be confusing. So if you'd rather just use the old notation you can, but be sure you are using the correct notation when entering your answers in LON-CAPA, especially when solving story problems that contain variables other than x.

Also, the shortcut for understanding the correct order of ordered pairs is to use the name of the function. If the function is $f(x)$, x is the input and $f(x)$ is the output, so the ordered pairs will be $(x, f(x))$. For a function such as $F(d)$, like we'll see in Example 10, d is the input and $F(d)$ is the output, so the ordered pairs would be set-up as $(d, F(d))$.

<u>Example 2:</u> Given the function values $f(-3) = 1$ and $f(3) = 2$, complete the following:

a. Write the function values as ordered pairs.

b. Use the ordered pairs from part a. to find the slope of the line that passes through those two points.

$$m = \frac{\Delta f(x)}{\Delta x}$$

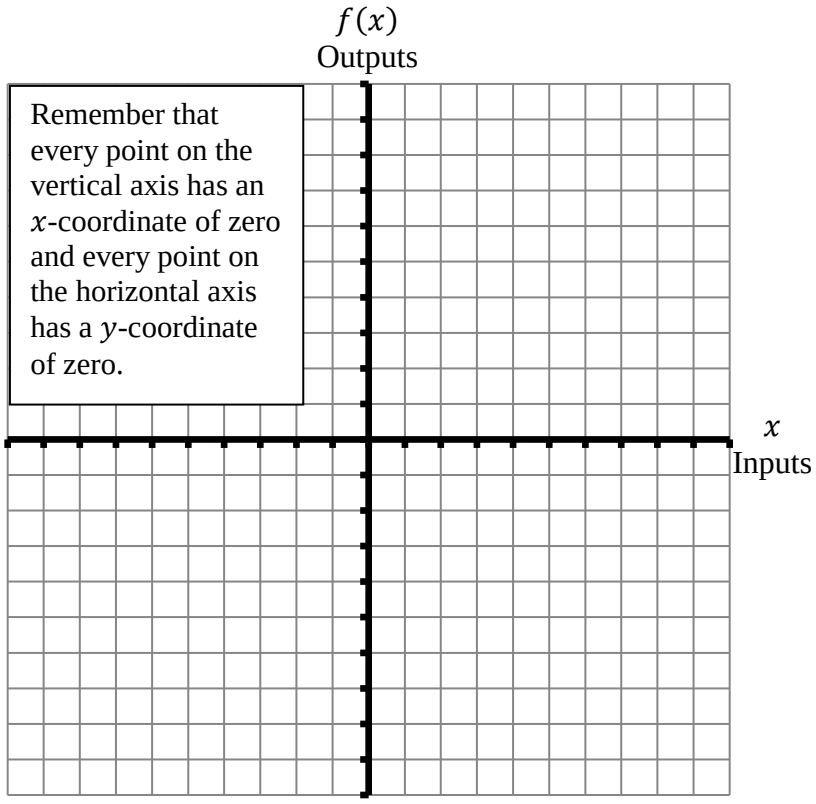
c. Write a linear function $f(x) = mx + b$ using the slope from part b. and one of the ordered pairs from part a.

$$f(x) - f(x_1) = m(x - x_1)$$

Example 3: Given the function values $f(-3) = 7$ and $f(0) = 0$, write a linear function $f(x) = mx + b$.

$f(x) - f(x_1) = m(x - x_1)$

Example 4: The graph of a linear function crosses the vertical axis at -7 and the horizontal axis at 2. Using that information, find the linear function $f(x) = mx + b$.



$f(x) - f(x_1) = m(x - x_1)$

Example 5: The depreciated value V of a machine is a linear function of time t. A machine is purchased for $\$120{,}000$, and will be worth $\$25{,}000$ in 8 years; write the linear function $V(t)$.

Just like on the previous problems, the first thing I will do is set-up ordered pairs. Keep in mind that the linear function we are finding is $V(t)$, where V represents the value of a machine based on time t. So time t will be the input of the function and value V will be the output. Therefore the ordered pairs will be set-up as $(\text{input}, \text{output})$ or $\big(t, V(t)\big)$.

The day the machine is purchased $(t = 0)$, the value of the machine is $\$120{,}000$. 8 years later $(t = 8)$, the value of the machine is $\$25{,}000$. So the ordered pairs will be $(0\,, 120000)$ and $(8, 25000)$.

Using those two ordered pairs, we can find the slope (or rate of change) of our linear function:

$$m = \frac{\Delta V}{\Delta t}$$

$$m = \frac{-95000}{8}$$

$$m = -11875$$

That means with every year that passes, the machine is work $\$11{,}875$ less than the year before.

Finally, we have two ordered pairs and the slope, so we can find the linear function by using point-slope form:

$$V(t) - V(t_1) = m(t - t_1)$$

$$V(t) - 120000 = -11875(t - 0)$$

$$\boldsymbol{V(t) = -11875t + 120000}$$

Since we know the initial value of the machine ($\$120{,}000$) and the rate of change (-11875), we could have also found our linear function without using point-slope form by simply plugging in the slope and the initial value to slope-intercept form, $\boldsymbol{V(t) = -11875t + 120000}$.

Example 6: A couple bought a house in 2005 for \$143,000; in 2015 they sold the house for \$179,000. If the value of the house H is a linear function of time t, where t represents the number of years since 2005, write the linear function $H(t)$.

$$H(t) - H(t_1) = m(t - t_1)$$

Example 7: A newborn baby weighed 7 pounds at birth and was 20 inches tall. After 5 months the baby weighs 16 pounds and is 25 inches tall. Write a linear function $W(t)$ to represent the baby's weight in pounds (W) at t months of age. Also write a linear function $H(t)$ to represent the baby's height in inches (H) at t months of age.

Since we need to find two linear functions, one for the babies height $H(t)$ and one for the babies weight $W(t)$, I've made a table to keep everything organized:

	$\boldsymbol{H(t)}$	$\boldsymbol{W(t)}$
Ordered Pairs	$(0, 20), (5, 25)$	
Slope	$m = \frac{\Delta H}{\Delta t} = \frac{5}{5} = 1$	
Linear Function	$H(t) = t + 20$	

a. What will the child's approximate weight and height be on their first birthday?

<u>Example 8:</u> A plumber charges \$50 for a service call plus \$80 per hour. Write a linear function that represents the plumbers total daily charges C (in dollars) as a function of time t (in hours).

To find the linear function for this problem, we can use the variable cost (\$80 per hour) as the rate of change, or slope, and the fixed cost (\$50 for a service call) as the initial value, or the b value in slope-intercept form $f(x) = mx + b$.

So in this case, our linear function will be $C(t) = 80t + 50$, since every customer must pay \$50 just to get a plumber to show up at their house, then they must pay \$80 per hour for every hour (or fraction of an hour) that the plumber is there.

$$\boldsymbol{C(t) = 80t + 50}$$

Keep in mind that this type of application problem could also be solved using ordered pairs like the previous problems, but using variable cost and fixed cost is much more efficient. Using ordered pairs, we'd have $(0, 50), (1, 50 + 80), (2, 50 + 80 + 80), \ldots$

Example 9: Bird scooters cost \$1 to unlock and \$0.20 per minute. Write a linear function which models the total cost for a ride C (in \$) as a function of the total time t (in minutes).

a. How much does it cost to ride for half an hour?

b. Is \$12 enough to ride for an hour?

We have two options to answer this question. One, we could replace t with 60 minutes (1 hour) to find the cost to ride for an hour, or two, we could replace $C(t)$ with 12 to find out how long we can ride for \$12.

Simplify $C(60) = 0.2(60) + 1$ Solve $12 = 0.2t + 1$ for t

Example 10: A bakery charges a \$2 delivery fee plus \$1.50 per cupcake, with a minimum of a dozen cupcakes required. Write a function that represents the total price P (in dollars) as a function of c (number of cupcakes).

a. How much does it cost to buy 20 cupcakes and have them delivered?

b. If I have \$20, how many cupcakes can I have delivered?

This chart is simply to show the connection between the linear equations that we covered in Lessons 16 & 17, and the linear functions that we covered in this lesson. Notice that the only real change going from equations to functions is replacing the variable y with the function notation $f(x)$.

	Linear Equations	**Linear functions**
Ordered Pairs	(x, y)	$\big(x, f(x)\big)$
Slope	$m = \dfrac{\Delta y}{\Delta x}$	$m = \dfrac{\Delta f(x)}{\Delta x}$
Point-slope form	$y - y_1 = m(x - x_1)$	$f(x) - f(x_1) = m(x - x_1)$
Slope-intercept form	$y = mx + b$	$f(x) = mx + b$

Answers to Examples:

1. $f(x) = 3x - 7$; *2.* $f(x) = \frac{1}{6}x + \frac{3}{2}$; *3.* $f(x) = -\frac{7}{3}x$;

4. $f(x) = \frac{7}{2}x - 7$; *5.* $V(t) = -11{,}875t + 120{,}000$;

6. $H(t) = 3{,}600t + 143{,}000$; *7.* $W(t) = \frac{9}{5}t + 7$; $H(t) = t + 20$;

7a. $W(12) = \frac{143}{5}\ lbs$; $H(12) = 32\ inches$; *8.* $C(t) = 80t + 50$;

9. $C(t) = 0.2t + 1$; *9a.* $7 ; *9b.* NO ;

10. $P(c) = 1.5C + 2$; *10a.* $P(20) = \$32$; *10b.* $c = 12$;