**Logarithmic Functions:**
- the inverse of an exponential function
  - the function that undoes an exponential function
  - if \( f(x) = a^x \), then \( f^{-1}(x) = \log_a(x) \)

\[
\begin{align*}
f(x) &= a^x & f^{-1}(x) &= \log_a(x) \\
\text{Domain: } (-\infty, \infty) & & \text{Domain: } (0, \infty) \\
\text{Range: } (0, \infty) & & \text{Range: } (-\infty, \infty)
\end{align*}
\]

Any real number can be the input of an exponential function \( f(x) = a^x \), but only positive outputs are possible because a positive number to the power of any real number is positive.

Only positive real numbers can be the argument of a logarithmic function \( g(x) = \log_a(x) \), but any real number is possible for the output because the output of a logarithmic function is the same as the input of its inverse (an exponential function), an exponent.

When \( f(x) = \log_a(x) \), the argument (the quantity in parentheses) is a power of the base \( a \) and the output is the exponent which produces that power. The argument must be a positive number, and the base \( a \) must be a positive number other than 1.

A logarithm is an exponent; a logarithm represents the exponent needed to change a base into a power of that base (the argument). When finding function values for a logarithmic function, think about what exponent is needed to make the base equal to the argument. For a function like \( f(x) = \log_3(x) \), we need to determine what exponent make 3 equal to \( x \).

\[
\begin{align*}
\log_3(9) &= 2 \text{ because } 3^2 = 9 \\
\log_3(1) &= 0 \text{ because } 3^0 = 1 \\
\log_3 \left( \frac{1}{3} \right) &= -1 \text{ because } 3^{-1} = \frac{1}{3}
\end{align*}
\]
Common logarithm:
- \( f(x) = \log(x) \)
  - when no base is denoted, it is understood to be base 10
  - \( f(x) = \log(x) \) is equivalent to \( f(x) = \log_{10}(x) \)

Just like any other logarithms, common logs are exponents. A common log is the exponent needed to make 10 become a power of 10

\[
\log(100) = 2 \text{ because } 10^2 = 100
\]

\[
\log \left( \frac{1}{10,000} \right) = -4 \text{ because } 10^{-4} = \frac{1}{10,000}
\]

Keep in mind that logarithms are simply exponents; when working with logarithms, ask yourself what exponent makes the base equal to the argument.

**Example 1:** Given the logarithmic function \( f(x) = \log(x) \), find the following function values. If no function value exists for a particular input, write DNE. *You may use your calculator to find these function values.*

a. \( f(10) \)
   \[
   \log_{10}(10) = 1 \text{ because } 10^1 = 10
   \]

b. \( f(100) \)
   \[
   \log_{10}(100) = 2 \text{ because } 10^2 = 100
   \]

c. \( f(1000) \)
   \[
   \log_{10}(1000) = 3 \text{ because } 10^3 = 1000
   \]

d. \( f \left( \frac{1}{10} \right) \)

 e. \( f(0.01) \)

 f. \( f \left( \frac{1}{1000} \right) \)

g. \( f(-10) \)

 h. \( f(1) \)

 i. \( f(0) \)

 j. \( f(0.0001) + f(1,000,000) - f \left( \frac{\sqrt[3]{100}}{1000} \right) + f(\sqrt{0.1}) \)
**Natural logarithm:**
- \( g(x) = \ln(x) \)
  - this is a logarithm with a base of \( e \)
- \( g(x) = \ln(x) \) is equivalent to \( g(x) = \log_e(x) \)
  - \( \ln \) means log base \( e \), so if you write \( \ln \), there is no need to include a base
  - also, do not write \( x \) (the input) as the base
- when you see \( g(x) = \ln(x) \), feel free to re-write the function as \( g(x) = \log_e(x) \)

**Example 2:** Given the logarithmic function \( g(x) = \ln(x) \), find the following function values. If no function value exists for a particular input, write DNE.  **You may use your calculator to find these function values.**

a. \( g(e) \)

\[ \log_e(e) = 1 \]

because \( e^1 = e \)

b. \( g\left(\frac{1}{e}\right) \)

\[ \log_e\left(\frac{1}{e}\right) = 2 \]

because \( e^{-1} = \frac{1}{e} \)

c. \( g\left(-\frac{1}{e}\right) \)

\[ \log_e\left(-\frac{1}{e}\right) \text{ DNE} \]

because \( e^x > 0 \)

d. \( g(e^2) \)

e. \( g\left(\frac{1}{e^2}\right) \)

f. \( g(-1) \)

g. \( g(1) \)

h. \( g(0) \)

i. \( g(e^5) \)

j. \( g\left(\sqrt{e}\right) - g\left(\frac{1}{\sqrt{e}}\right) + g(e^\pi) \)
As mentioned on page 1 of these notes, the domain of a logarithmic function is \((0, \infty)\) and the range is \((-\infty, \infty)\). The two logarithmic functions we worked with today (common logs and natural logs) are no different:

\[
\begin{align*}
    f(x) &= \log(x) \\
    g(x) &= \ln(x)
\end{align*}
\]

**Domain:** \((0, \infty)\)  
**Domain:** \((0, \infty)\)

The domains of \(f(x) = \log(x)\) and \(g(x) = \ln(x)\) are limited to positive numbers only because the input \(x\) is the argument of the logarithm, and the argument must be positive because it is either a power of 10 or a power of \(e\).

**Range:** \((-\infty, \infty)\)  
**Range:** \((0, \infty)\)

The ranges of \(f(x) = \log(x)\) and \(g(x) = \ln(x)\) are all real numbers because the output of a logarithmic function is an exponent, and an exponent can be any real number.

<table>
<thead>
<tr>
<th>Inputs</th>
<th>Outputs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>(f(x) = \log(x))</td>
</tr>
<tr>
<td>(\frac{1}{100})</td>
<td>(\log\left(\frac{1}{100}\right) = -2)</td>
</tr>
<tr>
<td>(\frac{1}{10})</td>
<td>(\log\left(\frac{1}{10}\right) = -1)</td>
</tr>
<tr>
<td>1</td>
<td>(\log(1) = 0)</td>
</tr>
<tr>
<td>10</td>
<td>(\log(10) = 1)</td>
</tr>
<tr>
<td>100</td>
<td>(\log(100) = 2)</td>
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</tr>
<tr>
<td>(\frac{1}{e})</td>
<td>(\ln\left(\frac{1}{e}\right) = -1)</td>
</tr>
<tr>
<td>1</td>
<td>(\ln(1) = 0)</td>
</tr>
<tr>
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</tr>
<tr>
<td>(e)</td>
<td>(\ln(e) = 1)</td>
</tr>
<tr>
<td>(e^2)</td>
<td>(\ln(e^2) = 2)</td>
</tr>
</tbody>
</table>

Keep in mind that just like any other function, logarithmic functions can be transformed, and transforming a log function could change its domain.
Answers to Examples:
1a. 1; 1b. 2; 1c. 3; 1d. −1; 1e. −2; 1f. −3; 1g. DNE;
1h. 0; 1i. DNE; 1j. \( \frac{5}{6} \);
2a. 1; 2b. −1; 2c. DNE; 2d. 2; 2e. −2; 2f. DNE; 2g. 0;
2h. DNE; 2i. 5; 2j. \( 1 + \pi \) ;