In this section we will be working with Properties of Logarithms in an attempt to take equations with more than one logarithm and condense them down into just a single logarithm.

**Properties of Logarithms:**

a. **Product Rule:**

\[ \log_c(a) + \log_c(b) = \log_c(a \cdot b) \]

*When two or more logarithms with the same base are added, those logarithms can be condensed into one logarithm whose argument is the product of the original arguments (a \cdot b in the example above)*

Order is not important when multiplying or adding, so changing the order of the factors in an argument or changing the order of the terms being added together does not change the answer.

b. **Quotient Rule:**

\[ \log_c(a) - \log_c(b) = \log_c \left( \frac{a}{b} \right) \]

*When two or more logarithms with the same base are subtracted, those logarithms can be condensed into one logarithm whose argument is the quotient of the original arguments \( \left( \frac{a}{b} \text{ in the example above} \right) \)*

Order is important when dividing and subtracting; the numerator is always the argument of the first log term listed (or the argument of the term listed first is always the numerator).

c. **Power Rule:**

\[ p \cdot \log_c(a) = \log_c(a^p) \]

*A factor times a logarithm can be re-written as the argument of the logarithm raised to the power of that factor*
Example 1: Solve each of the following logarithmic equations and **CHECK YOUR SOLUTIONS. LEAVE ANSWERS IN EXACT FORM, DO NOT APPROXIMATE**.

a. \(2 = \log_3(x) - \log_3(x - 1)\)  

\[
2 = \log_3 \left( \frac{x}{x-1} \right)
\]

\[
3^2 = \frac{x}{x-1}
\]

\[
9 = \frac{x}{x-1}
\]

\[
9(x - 1) = x
\]

\[
9x - 9 = x
\]

\[
8x = 9
\]

\[
x = \frac{9}{8}
\]

b. \(\log_2(x + 7) + \log_2(x) = 3\)  

\[
\log_2(x^2 + 7x) = 3
\]

\[
x^2 + 7x = 2^3
\]

\[
x^2 + 7x = 8
\]

\[
x^2 + 7x - 8 = 0
\]

\[
(x + 8)(x - 1) = 0
\]

\[
x = -8 \; ; \; x = 1
\]

Regardless of what type of answer you come up with (negative, positive, or zero), you must check your answer to verify that it results in a positive argument. To do so, **ALWAYS** plug your answer back into the original equation.

Replacing \(x\) with \(\frac{9}{8}\) will make each argument in the original equation positive, so \(x = \frac{9}{8}\) is a valid answer.

Replacing \(x\) with \(-8\) will make each argument in the original equation negative, so \(x = -8\) is not a valid answer. \(x = 1\) is a valid answer because it makes the arguments positive.

\[
x = \frac{9}{8}
\]

\[
x = 1
\]
In Example 1, the Properties of Logarithms were only used to combine logarithms in each problem (combining \( \log_3(x) \) with \( \log_3(x - 1) \) in part a., and \( \log_2(x + 7) \) with \( \log_2(x) \) in part b.). This is how we will be using the Properties of Logarithms in this class, to combine logarithms in order to reduce the number logarithms we have to just one, so that we can then convert that one logarithm to exponential form to solve.

If an equation has more than one logarithm on either side of the equation, use the Properties of Logarithms to **simplify as much as possible**, then solve by converting to exponential form. **REMEMBER THAT YOU MUST CHECK YOUR ANSWERS WHEN SOLVING LOG EQUATIONS TO VERIFY THAT THEY MAKE THE ORIGINAL ARGUMENTS POSITIVE.**

**Example 2:** Solve each of the following logarithmic equations and CHECK YOUR SOLUTIONS. **LEAVE ANSWERS IN EXACT FORM, DO NOT APPROXIMATE.**

a. \( 2 \ln(x) - \ln(5x - 6) = 0 \)  
b. \( \log(1 - x) = 1 - \log(x - 1) \)
c. \( \frac{1}{2} \log(x + 1) = \log(1 - x) \)

\[
\log(x + 1)^{\frac{1}{2}} = \log(1 - x)
\]

\[
\log(\sqrt{x + 1}) = \log(1 - x)
\]

\[
\log(\sqrt{x + 1}) - \log(1 - x) = 0
\]

\[
\log\left(\frac{\sqrt{x + 1}}{1-x}\right) = 0
\]

\[
\frac{\sqrt{x + 1}}{1-x} = 10^0
\]

\[
\frac{\sqrt{x + 1}}{1-x} = 1
\]

\[
\sqrt{x + 1} = 1 - x
\]

\[
(\sqrt{x + 1})^2 = (1 - x)^2
\]

\[
x + 1 = (1 - x)(1 - x)
\]

\[
x + 1 = 1 - 2x + x^2
\]

\[
0 = x^2 - 3x
\]

\[
0 = x(x - 3)
\]

\[
0 = x \; ; \; x - 3 = 0
\]

\[
x = 0 \; ; \; x = 3
\]

Replacing \( x \) with 0 in the original equation makes each argument positive, so \( x = 0 \) is a valid answer. Replacing \( x \) with 3 results in a negative argument \((1 - 3 = -2)\), so \( x = 3 \) is not a valid answer.

\[
x = 0
\]
**Example 3:** Solve each of the following logarithmic equations and **CHECK YOUR SOLUTIONS.** **LEAVE ANSWERS IN EXACT FORM, DO NOT APPROXIMATE.**

a. \( \log(x + 2) - \log(x) = 2 \log(4) \)

b. \( \log_{4}(3x + 2) = \log_{4}(5) + \log_{4}(3) \)
c. $\ln(x - 2) = 1 - \ln(x)$  (hint: when you get a quadratic equation, solve by completing the square rather than using the quadratic formula)

d. $\log_5(10) - \log_5(50) = \log_5(0.5) + \log_5(4x + 6)$
Keep in mind that while we have a Product Rule, Quotient Rule, and Power Rule for Logarithms, there is no Property of Logarithms that applies when the argument is a sum or difference.

\[
\log_c(a + b) \neq \log_c(a) + \log_c(b)
\]

\[
\log_c(a - b) \neq \log_c(a) - \log_c(b)
\]

Therefore equations such as \( \ln(x^2 + 1) = 2 \) or \( \log \left(x^2 - \frac{5}{4}\right) = 0 \) CANNOT be simplified using any Properties of Logarithms first before solving. Instead, these types of equations would simply be solved by converting to exponential form.

**Example 4:** Solve each of the following logarithmic equations and CHECK YOUR SOLUTIONS. LEAVE ANSWERS IN EXACT FORM, DO NOT APPROXIMATE. If there is more than one solution, separate them with commas. If there is no solution, enter NO SOLUTION.

a. \( \ln(x^2 + 1) = 2 \)  
b. \( \ln(x^2 + 1) = -1 \)
c. \( \log (x^2 - \frac{5}{4}) = 0 \) 

\hspace{2cm}
d. \( \log_5 \left( \frac{12}{x} - \frac{x}{2} \right) = 1 \)

\textbf{Answers to Examples:}

\begin{align*}
2a. & \quad x = 2; x = 3 \quad ; \quad 2b. \quad \text{NO SOLUTION} \quad ; \quad 2c. \quad x = 0 \quad ; \\
3a. & \quad x = \frac{2}{15} \quad ; \quad 3b. \quad x = \frac{13}{3} \quad ; \quad 3c. \quad x = 1 + \sqrt{e + 1} \quad ; \quad 3d. \quad x = -\frac{7}{5} \quad ; \\
4a. & \quad x = \pm \sqrt{e^2 - 1} \quad ; \quad 4b. \quad \text{NO SOLUTION} \quad ; \quad 4c. \quad x = \pm \frac{3}{2} \quad ; \\
4d. & \quad x = -12 \quad ; \quad x = 2 \quad ;
\end{align*}