In this section we will be working with Properties of Logarithms in an attempt to take equations with more than one logarithm and condense them down into just a single logarithm.

**Properties of Logarithms:**

a. **Product Rule:**

\[
\log_c(a) + \log_c(b) = \log_c(a \cdot b)
\]

When two or more logarithms with the same base are added, those logarithms can be condensed into one logarithm whose argument is the product of the original arguments (\(a \cdot b\) in the example above).

Order is not important when multiplying or adding, so changing the order of the factors in an argument or changing the order of the terms being added together does not change the answer.

b. **Quotient Rule:**

\[
\log_c(a) - \log_c(b) = \log_c\left(\frac{a}{b}\right)
\]

When two or more logarithms with the same base are subtracted, those logarithms can be condensed into one logarithm whose argument is the quotient of the original arguments \(\left(\frac{a}{b}\right)\) in the example above).

Order is important when dividing and subtracting; the numerator is always the argument of the first log term listed (or the argument of the term listed first is always the numerator).

c. **Power Rule:**

\[
p \cdot \log_c(a) = \log_c(a^p)
\]

A factor times a logarithm can be re-written as the argument of the logarithm raised to the power of that factor.
Example 1: Solve each of the following logarithmic equations and CHECK YOUR SOLUTIONS. LEAVE ANSWERS IN EXACT FORM, DO NOT APPROXIMATE.

a. \(2 = \log_3(x) - \log_3(x - 1)\)

\[
2 = \log_3 \left( \frac{x}{x-1} \right)
\]

\[
3^2 = \frac{x}{x-1}
\]

\[
9 = \frac{x}{x-1}
\]

\[
9(x - 1) = x
\]

\[
9x - 9 = x
\]

\[
8x = 9
\]

\[
x = \frac{9}{8}
\]

Regardless of what type of answer you come up with (negative, positive, or zero), you must check your answer to verify that it results in a positive argument. To do so, ALWAYS plug your answer back into the original equation.

Replacing \(x\) with \(\frac{9}{8}\) will make each argument in the original equation positive, so \(x = \frac{9}{8}\) is a valid answer.

b. \(\log_2(x + 7) + \log_2(x) = 3\)

\[
\log_2((x + 7)(x)) = 3
\]

\[
\log_2(x^2 + 7x) = 3
\]

\[
x^2 + 7x = 2^3
\]

\[
x^2 + 7x = 8
\]

\[
x^2 + 7x - 8 = 0
\]

\[
(x + 8)(x - 1) = 0
\]

\[
x = -8 \; ; \; x = 1
\]

Replacing \(x\) with \(-8\) will make each argument in the original equation negative, so \(x = -8\) is not a valid answer. \(x = 1\) is a valid answer because it makes the arguments positive.

\[
x = \frac{9}{8}
\]

\[
x = 1
\]
In Example 1, the Properties of Logarithms were only used to combine logarithms in each problem. This is how we will be using the Properties of Logarithms in this class, to combine logarithms in order to reduce the number logarithms we have to just one, so that we can then convert that one logarithm to exponential form to solve.

If an equation has more than one logarithm on either side of the equation, use the Properties of Logarithms to simplify as much as possible, then solve by converting to exponential form. Remember that you must check your answers when solving log equations to verify that they make the original arguments positive.

Example 2: Solve each of the following logarithmic equations and CHECK YOUR SOLUTIONS. LEAVE ANSWERS IN EXACT FORM, DO NOT APPROXIMATE.

a. $2 \ln(x) - \ln(5x - 6) = 0$ 

b. $\log(1 - x) = 1 - \log(x - 1)$
c. \( \frac{1}{2} \log(x + 1) = \log(1 - x) \)

\[
\log(x + 1)^{\frac{1}{2}} = \log(1 - x)
\]

\[
\log(\sqrt{x + 1}) = \log(1 - x)
\]

\[
\log(\sqrt{x + 1}) - \log(1 - x) = 0
\]

\[
\log \left( \frac{\sqrt{x} + 1}{1-x} \right) = 0
\]

\[
\frac{\sqrt{x}+1}{1-x} = 10^0
\]

\[
\frac{\sqrt{x}+1}{1-x} = 1
\]

\[
\sqrt{x} + 1 = 1 - x
\]

\[
(\sqrt{x} + 1)^2 = (1 - x)^2
\]

\[
x + 1 = (1 - x)(1 - x)
\]

\[
x + 1 = 1 - 2x + x^2
\]

\[
0 = x^2 - 3x
\]

\[
0 = x(x - 3)
\]

\[0 = x ; \quad x - 3 = 0
\]

\[x = 0 ; \quad x = 3\]

Replacing \( x \) with 0 in the original equation makes each argument positive, so \( x = 0 \) is a valid answer. Replacing \( x \) with 3 results in a negative argument \((1 - 3 = -2)\), so \( x = 3 \) is not a valid answer.

\[x = 0\]
Example 3: Solve each of the following logarithmic equations and CHECK YOUR SOLUTIONS. LEAVE ANSWERS IN EXACT FORM, DO NOT APPROXIMATE.

a. \( \log(x + 2) - \log(x) = 2\log(4) \)

b. \( \log_4(3x + 2) = \log_4(5) + \log_4(3) \)
c. \( \ln(x - 2) = 1 - \ln(x) \)

\[
\begin{align*}
\ln(x - 2) + \ln(x) &= 1 \\
\ln((x - 2)(x)) &= 1 \\
\log_e(x^2 - 2x) &= 1 \\
x^2 - 2x &= e^1 \\
x^2 - 2x + \left(\frac{-2}{2}\right)^2 &= e + \left(\frac{-2}{2}\right)^2 \\
x^2 - 2x + 1 &= e + 1 \\
(x - 1)^2 &= e + 1 \\
x - 1 &= \pm \sqrt{e + 1} \\
x &= 1 \pm \sqrt{e + 1} \\
x &= 1 + \sqrt{e + 1} ; \; x = 1 - \sqrt{e + 1}
\end{align*}
\]

\( x = 1 + \sqrt{e + 1} \) is approximately 2.928285 ..., which means if we plug this back into the original equation for \( x \), we’ll get positive arguments. This makes \( x = 1 + \sqrt{e + 1} \) a valid answer.

\( x = 1 - \sqrt{e + 1} \) is approximately -0.928285 ..., which means if we plug this back into the original equation for \( x \), we’ll get negative arguments. This makes \( x = 1 - \sqrt{e + 1} \) an invalid answer.

Keep in mind that while it may be necessary to approximate an answer in order to check it, you should always input exact answers only, unless LON-CAPA states otherwise.

\[ x = 1 - \sqrt{e + 1} \]
d. \( \log_5(10) - \log_5(50) = \log_5(0.5) + \log_5(4x + 6) \)

e. \( 2 \log_2(\sqrt{1-x}) - \log_2 \left( -\frac{1}{x} \right) + 5 = 7 - \log_2(9) \)
Keep in mind that while we have a Product Rule, Quotient Rule, and Power Rule for Logarithms, there is no Property of Logarithms that applies when the argument is a sum or difference.

\[
\log_c(a + b) \neq \log_c(a) + \log_c(b)
\]

\[
\log_c(a - b) \neq \log_c(a) - \log_c(b)
\]

Therefore equations such as \( \ln(x^2 + 1) = 2 \) or \( \log\left(x^2 - \frac{5}{4}\right) = 0 \) CANNOT be simplified using any Properties of Logarithms first before solving. Instead, these types of equations would simply be solved by converting to exponential form.

**Example 4:** Solve each of the following logarithmic equations and **CHECK YOUR SOLUTIONS. LEAVE ANSWERS IN EXACT FORM, DO NOT APPROXIMATE.** If there is more than one solution, separate them with commas. If there is no solution, enter NO SOLUTION.

a. \( \ln(x^2 + 1) = 2 \)  
b. \( \ln(x^2 + 1) = -1 \)
c. \( \log \left( x^2 - \frac{5}{4} \right) = 0 \)

\[
\log_{10} \left( x^2 - \frac{5}{4} \right) = 0
\]

\[x^2 - \frac{5}{4} = 10^0\]

\[x^2 - \frac{5}{4} = 1\]

\[4x^2 - 5 = 4\]

\[4x^2 = 9\]

\[x^2 = \frac{9}{4}\]

\[x = \pm \sqrt{\frac{9}{4}}\]

\[x = -\frac{3}{2} \; \text{or} \; x = \frac{3}{2}\]

Replacing \( x \) with \( -\frac{3}{2} \) or \( \frac{3}{2} \) in the argument of the original equation will result in a positive argument, so both \( -\frac{3}{2} \) and \( \frac{3}{2} \) are valid answers

\[x = -\frac{3}{2}, \frac{3}{2}\]

d. \( \log_5 \left( \frac{12}{x} - \frac{x}{2} \right) = 1 \)

\[
\log_5 \left( \frac{12}{x} - \frac{x}{2} \right) = 1
\]

\[
\frac{12}{x} - \frac{x}{2} = 5
\]

\[
\frac{12}{x} - \frac{x}{2} = 5
\]

\[24 - x^2 = 10x\]

\[0 = x^2 + 10x - 24\]

\[0 = (x + 12)(x - 2)\]

\[x = -12 \; \text{or} \; x = 2\]

Replacing \( x \) with \(-12\) or \( 2 \) in the argument of the original equation will result in a positive argument, so both \(-12\) and \( 2 \) are valid answers

\[x = -12, 2\]
Answers to Examples:

2a. \( x = 2; x = 3 \); 2b. NO SOLUTION; 2c. \( x = 0 \);

3a. \( x = \frac{2}{15} \); 3b. \( x = \frac{13}{3} \); 3c. \( x = 1 + \sqrt{e + 1} \); 3d. \( x = -\frac{7}{5} \);

3e. \( x = -\frac{1}{3} \); 4a. \( x = -\sqrt{e^2 - 1}, \sqrt{e^2 - 1} \); 4b. NO SOLUTION;

4c. \( x = -\frac{3}{2}, \frac{3}{2} \); 4d. \( x = -12; x = 2 \);

If you watched the video I recorded for this set of notes, you saw that I solved Example 3 part e and came up with a different answer than the one I originally had listed in these notes. I went through the problem once again to double check, and found that the answer I came up with in the video was correct, so I changed the answer here. That's why you may notice that the answer that is now listed here matches the answer that I found in the video. Sorry for any confusion that the original incorrect answer may have caused. As always, please email me (pdevlin@purdue.edu) with any questions.