**Product Rule for Radicals:**
- when indices are the same, radicands can be multiplied if all the roots exist
- this can be used to combine radicals or break them apart
  \[ \sqrt{3} \cdot \sqrt{7} = \sqrt{21} \]
  \[ \sqrt{27} = \sqrt{9 \cdot 3} = 3\sqrt{3} \]

- in some instances we will need to use the Product Rule to do both; combine two radicals first and then break them apart
  \[ \sqrt{6} \cdot \sqrt{15} = \sqrt{90} = \sqrt{9 \cdot 10} = 3\sqrt{10} \]
  \[ 3\sqrt{4} \cdot 3\sqrt{6} = 9\sqrt{24} = 9\sqrt{4 \cdot 6} = 9 \cdot 2\sqrt{6} = 18\sqrt{6} \]

**Quotient Rule for Radicals:**
- when indices are the same, radicands can be divided, if all the roots exist
- this can be used to combine radicals or break them apart
  \[ \frac{\sqrt{35}}{\sqrt{7}} = \sqrt{5} \]
  \[ \frac{\sqrt{2}}{\sqrt{9}} = \frac{\sqrt{2}}{3} \]
Simplifying Radicals:
- removing factors from the radical until no factor in the radicand has a degree greater than or equal to the index
  - \( \sqrt{x^9} \) is not simplified because the degree of the radicand (9) is larger than the index (2)
- use the Product (or Quotient) Rule for Radicals to simplify (as shown below on the left), or use fractional exponents (as shown on the right)
  - \( \sqrt{x^9} = x^{\frac{9}{2}} \)

Keep in mind that while 9 is a perfect square, \( x^9 \) is not. \( \sqrt{9} = 3 \) because \( 3^2 = 9 \). \( \sqrt{x^9} = x^4 \cdot \sqrt{x} \) because \( (x^4 \cdot \sqrt{x})^2 = x^9 \). Note that \( \sqrt{x^9} \) does not equal \( x^3 \) because \( (x^3)^2 = x^6 \), NOT \( x^9 \).

Also, when simplifying \( \sqrt{x^9} \) the long way, I wrote out \( \sqrt{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x} \) and then identified matching pairs of factors. The reason I was looking for pairs of factors is because the index of a square root is 2. Had I been trying to find the cubed root of \( x^9 \), I would have been looking for sets of 3 matching factors:

\[
\sqrt[3]{x^9} = \sqrt[3]{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x} = \sqrt[3]{(x \cdot x \cdot x) \cdot (x \cdot x \cdot x) \cdot (x \cdot x \cdot x)} = x \cdot x \cdot x = x^3
\]

The index of a radical always indicates how many common factors you’re looking for when simplifying.
Example 1: Simplify the following radical expressions completely. If you convert to fractional exponents, be sure to reduce those exponents as well (no improper fractions). Assume that all variables are positive.

a. $\sqrt[2]{9x^2y^4}$  

b. $\sqrt[3]{32a^8b^5}$

c. $\sqrt[3]{64x^{12}y^4}$  

d. $\sqrt[3]{16x^6y^8z^{16}}$
Be sure to keep in mind that simplifying radicals is a concept that students have had difficulty with on past exams. Be sure to spend enough time working on problems like these (in the notes, on the homework assignment, and when reviewing for the exam) so that you understand how to simplify radical expressions when you see them on the exam.

**Example 2:** Simplify the following radical expressions completely. If you convert to fractional exponents, be sure to reduce those exponents as well (no improper fractions). Assume that all variables are positive.

\[
a. \sqrt{2x^3y^5} \sqrt{8xy} \\
b. \sqrt{15xy^{-7}} \sqrt{10x^{-3}y^3}
\]

\[
a. \sqrt{150x^{-2}y^{-4}} \\
b. \frac{\sqrt{150}}{\sqrt{x^2y^4}} \\
\frac{\sqrt{150}}{\sqrt{x^2} \sqrt{y^4}} \\
\frac{\sqrt{25\sqrt{6}}}{x^1y^2} \\
\frac{5\sqrt{6}}{xy^2}
\]
c. $\sqrt[3]{3t^4v^2} \cdot \sqrt[3]{-9tv^5}$

\[3\sqrt{-27t^5v^7}\]

\[\sqrt[3]{27} \cdot \sqrt[3]{t^5} \cdot \sqrt[3]{v^7}\]

\[-3 \cdot \sqrt[3]{t^3} \cdot \sqrt[3]{t^2} \cdot \sqrt[3]{v^6} \cdot \sqrt[3]{v}\]

\[-3 \cdot t \cdot \sqrt[3]{t^2} \cdot \sqrt{v^2} \cdot \sqrt[3]{v}\]

\[-3t \sqrt{v^2} \cdot \sqrt[3]{t^2v}\]

We have seen the Product Rule for Radicals and the Quotient Rule for Radicals, and we have seen how to use them to simplify radicals. However, **there is no Sum Rule for Radicals or Difference Rule for Radicals:**

- $\sqrt{x^2 + y^2} = (x^2 + y^2)^{\frac{1}{2}}$, **BUT** $\sqrt{x^2 + y^2} \neq x + y$
- $\sqrt{a^2 - b^2} = (a^2 - b^2)^{\frac{1}{2}}$, **BUT** $\sqrt{a^2 - b^2} \neq a - b$
- when a radicand is a sum or difference, the radical cannot be distributed to each term
  - the radical can be re-written using a fractional exponent, but that fractional exponent **CANNOT** be distributed
- just as we did in Lesson 2, feel free to use numbers to show that there is no Sum Rule for Radicals or Difference Rule for Radicals
  - $\sqrt{3^2 + 4^2} \neq 3 + 4$
  - $\sqrt{5^2 - 4^2} \neq 5 - 4$

\[\sqrt{9 + 16} \neq 7\]

\[\sqrt{25} \neq 7\]

\[5 \neq 7\]

\[\sqrt{9} \neq 1\]

\[3 \neq 1\]
So with an expression such as \((25 - x)^{\frac{3}{2}}\), the power of \(\frac{3}{2}\) CANNOT be distributed to the terms 25 and \(x\). However we can reduce the fractional exponent of \(\frac{3}{2}\) just as we have with other fractional exponents before.

\[
- (25 - x)^{\frac{3}{2}} = \sqrt{(25 - x)^3} = \sqrt{25 - x} \cdot \sqrt{25 - x} = (25 - x) \cdot \sqrt{25 - x}
\]

Keep in mind that parentheses are necessary in this answer; \((25 - x) \sqrt{25 - x}\) and \(25 - x \sqrt{25 - x}\) are NOT equivalent.

**Example 3:** Simplify the following expressions completely. Be sure to reduce fractional exponents so they are not improper fractions (the numerator must be less than the dominator). Assume that all variables are positive.

a. \((4 + x)^{\frac{7}{2}}\)

\[
\sqrt{(4 + x)^7} = (4 + x)^{\frac{7}{2}} + x^{\frac{7}{2}}
\]

b. \(4^{\frac{7}{2}} + x^{\frac{7}{2}}\)

\[
\sqrt{(4 + x)(4 + x)(4 + x)(4 + x)(4 + x)(4 + x)(4 + x)} = \sqrt{(4 + x)(4 + x)(4 + x)\sqrt{4 + x}} = (4 + x)^{\frac{3}{2}} \sqrt{4 + x}
\]

\[
(4 + x)^{\frac{7}{2}} + x^{\frac{7}{2}} + x^{\frac{7}{2}} = (4 + x)^{\frac{7}{2}} + x^{\frac{7}{2}} + x^{\frac{7}{2}} = 128 + x^3 \sqrt{x}
\]

Notice that \((4 + x)^{\frac{7}{2}}\) and \(4^{\frac{7}{2}} + x^{\frac{7}{2}}\) are NOT equivalent. This once again shows that exponents can be distributed to sums or differences.
c. \((9 - x)^{\frac{5}{2}}\)  

d. \(9 - x^\frac{5}{2}\)

\[e. (8 + x)^{\frac{3}{2}}\]  

f. \(8^\frac{5}{3} + x\)

Once again, simplifying radicals is a concept that students have struggled with in prior semesters. Take your time when working through these problems in LON-CAPA, get help from SI, office hours, or other resources on campus if needed, and be sure you are prepare to simplify expressions like these when you see them on the exam.
Answers to Examples:

1a. $3xy^2$ ; 1b. $4a^4b^2 \cdot \sqrt{2b}$ ; 1c. $4x^3y^3\sqrt{y}$ ; 1d. $2x^2y^2z^5 \cdot \sqrt[3]{2y^2z}$

2a. $4x^2y^3$ ; 2b. $\frac{5\sqrt{6}}{xy^2}$ ; 2c. $-3tv^2 \cdot \sqrt[3]{t^2v}$ ; 2d. $5ab^2c^3 \cdot \frac{3\sqrt[3]{2a^2b^2c}}{x^2}$

3a. $(4 + x)^3 \cdot \sqrt{4 + x}$ ; 3b. $128 + x^3 \cdot \sqrt{x}$ ; 3c. $(9 - x)^2 \sqrt{9 - x}$

3d. $9 - x^2 \sqrt{x}$ ; 3e. $(8 + x) \sqrt{8 + x}$ ; 3f. $32 + x$