The following polynomial has four terms:

\[ xy + 2y + 3x + 6 \]

Notice that there is no common factor among the four terms (no GCF). However the first two terms do have a common factor of \( y \) and the last two terms have a common factor of 3. So while we can’t factor the polynomial by taking out a GCF, we can factor by grouping. This means grouping the first two terms and factoring out a GCF, then grouping the last two terms and factoring out a GCF.

\[ xy + 2y + 3x + 6 \]

\[ y(x + 2) + 3(x + 2) \]

We now have a sum of two terms, and both terms have a common factor of \( (x + 2) \). If we take out the GCF of \( (x + 2) \) we are left with the following:

\[ y(x + 2) + 3(x + 2) \]

\[ (x + 2) \left( \frac{y(x + 2)}{x + 2} + \frac{3(x + 2)}{x + 2} \right) \]

\[ (x + 2) \left( \frac{y \cdot 1}{1} + \frac{3 \cdot 1}{1} \right) \]

\[ (x + 2)(y + 3) \]

This is an example of factoring a polynomial by grouping the terms.
Factor by grouping:
- grouping the terms of a polynomial and factoring out a GCF from each group
- you can group any terms that have a common factor
  - this means the order of the two middle terms can be reversed and the final factored answer will remain the same; this will be important on the next page when we use the \(ac\)-method to factor

\[
\begin{align*}
9x^3 + 36x^2 + 4x + 16 & = 9x^3 + 4x + 36x^2 + 16 \\
9x^3 + 36x^2 + 4x + 16 & = 9x^3 + 4x + 36x^2 + 16 \\
9x^2(x + 4) + 4(x + 4) & = x(9x^2 + 4) + 4(9x^2 + 4) \\
(x + 4)(9x^2 + 4) & = (9x^2 + 4)(x + 4)
\end{align*}
\]

Example 1: Factor the following polynomials by grouping.

a. \(x^3 - 4x^2 + 6x - 24\) 
   
   Always check to see if the terms in the polynomial have a GCF. In this problem, the four terms do not have a GCF, so I will simply factor by grouping.
   
   \[x^2(x - 4) + 6(x - 4)\]
   
   \[(x - 4)(x^2 + 6)\]
We have seen how to factor polynomials that contain a GCF, and how to factor polynomials where only certain groups of terms have a GCF. Next we will look at an algorithm for factoring quadratic trinomials (trinomials with a degree of 2, such as $12x^2 + 17x - 5$). In Lesson 7 we’ll see examples of non-quadratic trinomials, such as $10x^6 - 13x^3 + 3$, and show how this algorithm can be used to factor those trinomials as well.

**Factoring trinomials using the ac-method** ($ax^2 + bx + c$):

1. check for a GCF first
   a. this should be done regardless of how you are factoring or what type of polynomial you have (binomial, trinomial …)
2. find two numbers whose product is $ac$ and whose sum is $b$
   a. $a$ is the leading coefficient of the polynomial and $c$ is the constant term, while $b$ is the coefficient of $x$
3. replace the middle term of the original trinomial ($bx$) with an expression containing the two numbers from step 2
4. factor the resulting polynomial by grouping

**Example 2:** Factor the following polynomials completely.
   a. $12x^2 + 17x - 5$
b. $6x^4 + 5x^3 + x^2$

c. $16x^2 + 24x + 9$

To factor this trinomial using the $ac$-method, I would start by trying to find two numbers with a product of $ac$ ($16 \cdot 9 = 144$) and a sum of $b$ ($24$). In this case, those two numbers are $12$ and $12$, so I will replace $24x$ with $12x + 12x$ in order to then factor by grouping.

$$16x^2 + 12x + 12x + 9$$

$$4x(4x + 3) + 3(4x + 3)$$

$$(4x + 3)(4x + 3)$$

Since I end up with the same binomial twice, I can express it as a perfect square.

$$(4x + 3)^2$$
When to factor out a negative factor rather than a positive factor:

There are two scenarios in which it is beneficial to factor out a negative factor rather than a positive factor:

1. When the leading coefficient is negative

\[ 18 + 15x - 3x^2 \]
\[ -3x^2 + 15x + 18 \]
\[ -3(x^2 - 5x - 6) \]

The trinomial \( x^2 - 5x - 6 \) should be easier to factor than \( -x^2 + 5x + 6 \), which we would have had if we’d factored out 3 instead of \(-3\).

2. To make the binomials match when using the \(ac\)-method

\[ -3(x^2 - 5x - 6) \]
\[ -3(x^2 + x - 6x - 6) \]
\[ -3(x(x + 1) - 6(x + 1)) \]
\[ -3(x + 1)(x - 6) \]

Had I factored out 6 from \(-6x - 6\), I would have been left with the binomial \((-x - 1)\) which would not have matched \((x + 1)\). By factoring out a \(-6\) instead, I had a common factor of \((x + 1)\), which I was then able to factor out.
Example 3: Factor the following polynomials completely.

a. $150 - 25x - x^2$

b. $-3x^3 + 17x^2 - 20x$

Since this trinomial has a negative leading coefficient, I will start by factoring out a negative GCF in order to make the leading coefficient positive.

$$-x(3x^2 - 17x + 20)$$

Next I will factor the trinomial that remains using the ac-method. To do so I will find two numbers with a of product $ac$ ($3 \cdot 20 = 60$) and a sum of $b$ ($-17$). In this case, those two numbers are $-5$ and $-12$, so I would replace $-17x$ with $-5x - 12x$ in order to then factor by grouping.

$$-x(3x^2 - 5x - 12x + 20)$$

$$-x(x(3x - 5) - 4(3x - 5))$$

$$-x(3x - 5)(x - 4)$$
Answers to Examples:

1a. \((x - 4)(x^2 + 6)\); 1b. \(2(4x - 1)(3x^2 + 1)\);
2a. \((3x + 5)(4x - 1)\); 2b. \(x^2(3x + 1)(2x + 1)\); 2c. \((4x + 3)^2\);
3a. \(-1(x - 5)(x + 30)\); 3b. \(-x(3x - 5)(x - 4)\)