Complex fraction:

- a fraction which has rational expressions in the numerator and/or denominator
  - $\dfrac{\frac{1}{x}-\frac{1}{2}}{x^2-4}$
  - $\dfrac{\frac{x}{y^2}+\frac{y}{x^2}}{\frac{1}{y^2}-\frac{1}{x^2}}$

Steps for Simplifying Complex Fractions

1. simplify the numerator and/or the denominator by adding and/or subtracting the rational expressions
2. use the procedure for dividing fractions to change division to multiplication
3. factor the numerator and denominator completely
4. simplify the fraction completely by canceling common factors

$$\frac{\frac{x}{y^2}+\frac{y}{x^2}}{\frac{1}{y^2}-\frac{1}{x^2}}$$

$$\frac{\frac{x^3+y^3}{x^2y^2}}{\frac{x^2-y^2}{x^2y^2}}$$

$$\frac{x^3+y^3}{x^2y^2} \div \frac{x^2-y^2}{x^2y^2}$$

$$\frac{x^3+y^3}{x^2y^2} \cdot \frac{x^2y^2}{x^2-y^2}$$

$$\frac{(x+y)(x^2-xy+y^2)(x^2y^2)}{(x^2y^2)(x-y)(x+y)}$$

$$\frac{x^2-xy+y^2}{x-y}$$

Simplifying complex fractions is basically just a combination of the concepts from the previous three lessons.

The rational expressions in the numerator and/or denominator of the complex fraction need to be added or subtracted first (Lesson 8).

Then the complex fraction gets converted to two rational expressions being divided, which we don't actually divide at all, we simply convert to multiplication by taking the reciprocal of the divisor (Lesson 8).

When two fractions are being multiplied, we write numerator times numerator and denominator times denominator, but we don't actually multiply them because we need to cancel common factors (Lesson 7). Once we factor completely, we can simplify the rational expressions by canceling common factors (Lesson 7). Remember that the numerator and denominator of the rational expression need to be factored completely in order to simplify, and factoring we covered in Lessons 6 and 7.

The ability to synthesize all the information from the previous three lessons is imperative to being able to simplify complex fractions.

It is imperative that you understand how to simplify, multiply, divide, add, and subtract rational expressions, as well as how to factor polynomials, in order to simplify complex fractions. All of those concepts are combined into one problem when simplifying complex fractions, and the inability to perform any of those tasks will prevent you from correctly answering these types of problems.

Example 1: Perform the indicated operations and simplify the expressions completely.

a. $$\frac{\frac{1}{x}-\frac{1}{2}}{x^2-4}$$

$$\frac{\frac{2}{2x}-\frac{x}{2x}}{x^2-4}$$

$$\frac{\frac{2-x}{2x}}{x^2-4}$$

$$\frac{2-x}{2x}\div(x^2-4)$$

$$\frac{2-x}{2x}\cdot\frac{1}{x^2-4}$$

$$\frac{2-x}{2x(x^2-4)}$$

$$\frac{-1(x-2)}{2x(x-2)(x+2)}$$

$$\mathbf{\frac{-1}{2x(x+2)}}$$

b. $$\frac{\frac{x}{y^2}-\frac{y}{x^2}}{\frac{1}{y^2}-\frac{1}{x^2}}$$

c. $\dfrac{\frac{y}{x}-\frac{x}{4y}}{\frac{x}{y}-2}$

d. $\dfrac{1+\left(\frac{x}{y}\right)^3}{\frac{x}{y}+1}$

$$\frac{1+\frac{x^3}{y^3}}{\frac{x}{y}+\frac{y}{y}}$$

$$\frac{\frac{y^3}{y^3}+\frac{x^3}{y^3}}{\frac{x+y}{y}}$$

$$\frac{\frac{y^3+x^3}{y^3}}{\frac{x+y}{y}}$$

$$\frac{y^3+x^3}{y^3} \div \frac{x+y}{y}$$

$$\frac{y^3+x^3}{y^3} \cdot \frac{y}{x+y}$$

$$\frac{(y^3+x^3)y}{y^3(x+y)}$$

$$\frac{(y+x)(y^2-xy+x^2)y}{y^3(x+y)}$$

$$\frac{\cancel{(y+x)}(y^2-xy+x^2)\cdot\cancel{y}}{y^2\cdot\cancel{y}\cdot\cancel{(x+y)}}$$

$$\boldsymbol{\frac{y^2-xy+x^2}{y^2}}$$

e. $\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$

f. $\dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$

$\dfrac{\frac{x^2}{x^2}\cdot\frac{1}{(x+h)^2}-\frac{1}{x^2}\cdot\frac{(x+h)^2}{(x+h)^2}}{h}$

$\dfrac{\frac{x^2}{x^2(x+h)^2}-\frac{(x+h)^2}{x^2(x+h)^2}}{h}$

$\dfrac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h}$

$\dfrac{\frac{x^2-(x+h)(x+h)}{x^2(x+h)^2}}{h}$

$\dfrac{\frac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2}}{h}$

$\dfrac{\frac{x^2-x^2-2xh-h^2}{x^2(x+h)^2}}{h}$

$\dfrac{\frac{-2xh-h^2}{x^2(x+h)^2}}{h}$

$\dfrac{-2xh-h^2}{x^2(x+h)^2}\div h$

$\dfrac{-2xh-h^2}{x^2(x+h)^2}\cdot\dfrac{1}{h}$

$\dfrac{h(-2x-h)}{h\cdot x^2(x+h)^2}$

$\dfrac{\cancel{h}(-2x-h)}{\cancel{h}\cdot x^2(x+h)^2}$

$\boldsymbol{\dfrac{-2x-h}{x^2(x+h)^2}}$

Negative Exponent Rule:

- once again, to change the sign of an **exponent**, take the reciprocal of the factor or expression that has the negative exponent
  - $x^{-3} = \frac{1}{x^3}$
  - $5x^{-3} = \frac{5}{x^3}$
  - $(5x)^{-2} = \frac{1}{25x^2}$

Example 2: Perform the indicated operations and simplify the following expression completely. Do not include negative exponents in your answer.

$\dfrac{(25x)^{-1} - x}{25x^{-1} - x}$		$\left(\dfrac{1-25x^2}{25x}\right) \div \left(\dfrac{25-x^2}{x}\right)$
↓		↓
$\dfrac{\frac{1}{25x} - x}{\frac{25}{x} - x}$		$\left(\dfrac{1-25x^2}{25x}\right) \cdot \left(\dfrac{x}{25-x^2}\right)$
↓		↓
$\dfrac{\frac{1}{25x} - \frac{x}{1} \cdot \frac{25x}{25x}}{\frac{25}{x} - \frac{x}{1} \cdot \frac{x}{x}}$		$\dfrac{(1-25x^2)(x)}{(25x)(25-x^2)}$
↓		↓
$\dfrac{\frac{1}{25x} - \frac{25x^2}{25x}}{\frac{25}{x} - \frac{x^2}{x}}$		$\dfrac{(1-5x)(1+5x)(x)}{(25)(x)(5-x)(5+x)}$
↓		↓
$\dfrac{\frac{1-25x^2}{25x}}{\frac{25-x^2}{x}}$		$\boldsymbol{\dfrac{(1-5x)(1+5x)}{25(5-x)(5+x)}}$

Re-arranging the terms or factors in an answer has no effect on that answer, so even if you express the answer as $\frac{(5x+1)(1-5x)}{25(x+5)(5-x)}$, the answer is still the same and it still does not simplify any further.

Also, keep in mind that in the original expression $\frac{(25x)^{-1} - x}{25x^{-1} - x}$ you cannot simply move $(25x)^{-1}$ from the numerator to the denominator to change the sign of the exponent. Same with $25x^{-1}$; you must take the reciprocal to change the sign of the exponent. Moving from numerator to denominator or vice versa is a shortcut that only works for factors, **NOT** terms.

Example 2: Perform the indicated operations and simplify the expressions completely. Do not include negative exponents in your answer.

a. $\frac{3y^{-1} - 3x^{-1}}{x^{-2} - y^{-2}}$

b. $\frac{xy^{-1} - (xy)^{-1}}{x^{-2} - 1}$

$$\frac{\frac{x}{y} - \frac{1}{xy}}{\frac{1}{x^2} - \frac{1}{1}}$$

$$\frac{\frac{x}{x}\cdot\frac{x}{y} - \frac{1}{xy}}{\frac{1}{x^2} - \frac{1}{1}\cdot\frac{x^2}{x^2}}$$

$$\frac{\frac{x^2 - 1}{xy}}{\frac{1 - x^2}{x^2}}$$

$$\frac{x^2 - 1}{xy} \div \frac{1 - x^2}{x^2}$$

$$\frac{x^2 - 1}{xy} \cdot \frac{x^2}{1 - x^2}$$

$$\frac{(-1)(1-x)(x+1)(x)(x)}{(x)(y)(1-x)(1+x)}$$

$$-\frac{x}{y}$$

c. $\dfrac{x^{-2}-(x-1)^{-2}}{(x-1)^2}$

$$\frac{\frac{1}{x^2}-\frac{1}{(x-1)^2}}{\frac{1}{(x-1)^2}}$$

$$\frac{\frac{(x-1)^2}{(x-1)^2}\cdot\frac{1}{x^2}-\frac{1}{(x-1)^2}\cdot\frac{x^2}{x^2}}{\frac{1}{(x-1)^2}}$$

$$\frac{\frac{(x-1)^2-x^2}{x^2(x-1)^2}}{\frac{1}{(x-1)^2}}$$

$$\frac{\frac{(x-1)(x-1)-x^2}{x^2(x-1)^2}}{\frac{1}{(x-1)^2}}$$

$$\frac{\frac{x^2-2x+1-x^2}{x^2(x-1)^2}}{\frac{1}{(x-1)^2}}$$

$$\frac{\frac{1-2x}{x^2(x-1)^2}}{\frac{1}{(x-1)^2}}$$

$$\frac{1-2x}{x^2(x-1)^2}\div\frac{1}{(x-1)^2}$$

$$\frac{1-2x}{x^2(x-1)^2}\cdot\frac{(x-1)^2}{1}$$

$$\frac{(1-2x)(x-1)^2}{x^2(x-1)^2}$$

$$\boldsymbol{\frac{1-2x}{x^2}}$$

d. $\dfrac{1-6x^{-1}+9x^{-2}}{x^{-2}-3^{-2}}$

Answers to Examples:

1a. $-\frac{1}{2x(x+2)}$ *; 1b.* $\frac{x^2+xy+y^2}{x+y}$ *; 1c.* $-\frac{2y+x}{4x}$ *; 1d.* $\frac{x^2-xy+y^2}{y^2}$ *;*

1e. $-\frac{1}{x(x+h)}$ *; 1f.* $-\frac{2x+h}{x^2(x+h)^2}$ *; 2a.* $\frac{-3xy}{x+y}$ *; 2b.* $-\frac{x}{y}$ *; 2c.* $\frac{1-2x}{x^2}$ *;*

2d. $\frac{-9(x-3)}{x+3}$ *;*