Chapter 3 is about second order linear diff eqs, i.e. those of the form
\[ y'' + p(t)y' + q(t)y = g(t) \]
Such an equation is called **homogeneous** if \( g(t) \equiv 0 \) (i.e. \( q \) is the zero function). If \( g(t) \not\equiv 0 \), then \( g(t) \) is called the **nonhomogeneous part** of the equation.

In this section, we deal with homogeneous equations in which the coefficient functions are constant, i.e. of the form \( ay'' + by' + cy = 0 \).

We'll motivate how to solve such equations with the example \( y'' - y = 0 \). It's not hard to see \( y_1(t) = e^t \) and \( y_2(t) = e^{-t} \) are both solutions. And for any constants \( C_1 \) and \( C_2 \), \( C_1 e^t + C_2 e^{-t} \) is a solution.

Thus, it seems natural that all solutions of \( ay'' + by' + cy = 0 \) will be of the form \( e^{rt} \) and if there's more than one, we get linear combinations.

Assuming \( y = e^{rt} \) is a solution of \( ay'' + by' + cy = 0 \), we see \( y'(t) = re^{rt} \), \( y''(t) = r^2 e^{rt} \), so the equation becomes
\[ ar^2 e^{rt} + br e^{rt} + ce^{rt} = 0 \]
\[ e^{rt} (ar^2 + br + c) = 0 \]
This is only possible when \( ar^2 + br + c = 0 \). Given a diff eq of the form \( ay'' + by' + cy = 0 \), the equation \( ar^2 + br + c = 0 \) is called the **characteristic equation** of the diff eq.

The roots of the characteristic polynomial satisfy the diff eq.

From algebra, we know the roots can be
(i) distinct real numbers (this lesson)
(ii) complex conjugates (Lesson 12)
(iii) a single repeated real number (Lesson 13)

Thus, given \( ay'' + by' + cy = 0 \):
- Find the roots of the characteristic polynomial.
- If they are real numbers \( r_1, r_2 \), then the general solution is of the form \( y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \) for \( c_1, c_2 \) constants.

**EX.1:** Find the general solution to \( y'' + y' - 12y = 0 \)

Characteristic equation: \( r^2 + r - 12 = 0 \)
\((r+4)(r-3) = 0 \) \( \Rightarrow r_1 = -4, r_2 = 3 \)

Gives solution: \( y(t) = c_1 e^{-4t} + c_2 e^{3t} \)

For second order equations, we require two initial conditions to find a particular solution.
**EX2:** Solve the IVP \( y'' + y' - 2y = 0, \ y(0) = 0, \ y'(0) = 3 \)

\[
r^2 \cdot r - 2 = 0 \Rightarrow (r-1)(r+2) = 0 \Rightarrow r = 1 \text{ or } -2
\]

\[
y(t) = c_1 e^t + c_2 e^{-2t}
\]

\[
y'(t) = c_1 e^t - 2c_2 e^{-2t}
\]

\[
\begin{bmatrix}
0 = y(0) = c_1 + c_2 \\
3 = y'(0) = c_1 - 2c_2
\end{bmatrix}
\]

gives a system of equations

\[
\begin{align*}
(1) \quad c_1 + c_2 &= 0 \\
(2) \quad c_1 - 2c_2 &= 3
\end{align*}
\]

\[
(1) - (2) = 0 - 3 = c_2 \quad \Rightarrow 3c_2 = -3 \quad \Rightarrow c_2 = -1
\]

\[
c_1 + 1 = 0 \quad \Rightarrow c_1 = 1
\]

\[
\Rightarrow y = e^t - e^{-2t}
\]

**EX3:** Determine the behavior of \( y(t) = c_1 e^{rt} + c_2 e^{r_2 t} \) based on the values of \( r_1 \) and \( r_2 \).

Note if \( r \) is negative, \( \lim_{t \to \infty} c e^{rt} = 0 \). If \( r \) is positive, \( \lim_{t \to \infty} c e^{rt} = \pm \infty \)

- If both \( r_1 \) and \( r_2 \) are negative, \( \lim_{t \to \infty} (c_1 e^{r_1 t} + c_2 e^{r_2 t}) = 0 \).
- If both \( r_1 \) and \( r_2 \) are positive, \( \lim_{t \to \infty} (c_1 e^{r_1 t} + c_2 e^{r_2 t}) = \pm \infty \), depending on the signs of \( c_1, c_2 \) and whether \( r_1 < r_2 \) or \( r_2 < r_1 \).
- If \( r_1 \) is positive and \( r_2 \) is non-positive, \( \lim_{t \to \infty} (c_1 e^{r_1 t} + c_2 e^{r_2 t}) = \pm \infty \), depending on the sign of \( c_1 \). In this case, we can have a max or min.
- If \( r_1 = 0 \) and \( r_2 \) is negative, \( \lim_{t \to \infty} (c_1 e^{0 t} + c_2 e^{r_2 t}) = c_2 \).

These assume \( c_1 \neq 0 \) and \( c_2 \neq 0 \).

**EX4:** Solve the IVP \( y'' - y' - 2y = 0, \ y(0) = \alpha, \ y'(0) = 2 \). Determine the value of \( \alpha \) so the solution approaches zero as \( t \to \infty \).

\[
r^2 - r - 2 = 0 \Rightarrow (r-2)(r+1) = 0 \Rightarrow r = 2, -1
\]

\[
y(t) = c_1 e^{2t} + c_2 e^{-t}
\]

\[
y'(t) = 2c_1 e^{2t} - c_2 e^{-t}
\]

\[
\begin{bmatrix}
\alpha = c_1 + c_2 \\
2 = 2c_1 - c_2
\end{bmatrix}
\]

\[
\Rightarrow 2 + \alpha = 3c_1 \quad \Rightarrow \frac{2 + \alpha}{3} = c_1
\]

\[
\alpha = \frac{2 + \alpha}{3} + c_2 \quad \Rightarrow -2 + 2\alpha = c_2
\]

\[
y(t) = \left(\frac{2 + \alpha}{3}\right) e^{2t} + \left(\frac{2\alpha - 2}{3}\right) e^{-t}
\]

\[
\Rightarrow \text{as } t \to \infty
\]

\[
\Rightarrow \text{need } \alpha = -2 \text{ to zero this term out.}
\]
EX5: Find the maximum value to the solution of \(2y'' - 3y' + y = 0\), \(y(0) = 2\), \(y'(0) = \frac{1}{2}\).

\[2r^2 - 3r + 1 = 0 \Rightarrow (2r - 1)(r - 1) = 0 \Rightarrow r = \frac{1}{2} \text{ or } 1.
\]

\[y(t) = c_1 e^{t/2} + c_2 e^t, \quad y'(t) = \frac{c_1}{2} e^{t/2} + c_2 e^t
\]

\[
\begin{bmatrix}
\frac{2}{2} = c_1 + c_2 \\
\frac{1}{2} = \frac{c_1}{2} + c_2
\end{bmatrix} \Rightarrow \frac{3}{2} = \frac{c_1}{2} \Rightarrow c_1 = 3
\]

\[2 = 3 + c_2 \Rightarrow c_2 = -1
\]

\[y = 3e^{t/2} - e^t
\]

\[y' = \frac{3}{2} e^{t/2} - e^t. \text{ Set } y' = 0 \text{ to find max!}
\]

\[\frac{3}{2} e^{t/2} - e^t = 0
\]

\[\frac{3}{2} - e^{-t/2} = 0 \quad \text{(multiplied by } e^{-t/2})
\]

\[\frac{3}{2} = e^{t/2} \Rightarrow \ln(\frac{3}{2}) = \frac{t}{2}
\]

\[\Rightarrow 2\ln(\frac{3}{2}) = t
\]

\[y(\ln(\frac{3}{2})) = 3e^{\ln(\frac{3}{2})/2} - e^{\ln(\frac{3}{2})} = -\ln(\frac{3}{2})
\]

\[= 3\left(\frac{3}{2}\right) - \frac{3}{2} = \frac{9}{2} - \frac{9}{4} = \frac{9}{4}.
\]