Recall from linear algebra that a function $L : V \to W$ between vector spaces is called **linear** if $L(v_1 + v_2) = L(v_1) + L(v_2)$ and $L(cv) = cL(v)$. A **linear operator** is a linear map for which our vector spaces have functions as their vectors.

Given a differential equation $y'' + py' + qy = 0$, we develop the linear operator $L[\phi] = \phi'' + p\phi' + q\phi$.

**Notice:** $L[\phi]$ is a function!

$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t)$

$\phi$ is a solution to $y'' + py' + qy = 0$ if and only if $L[\phi] = 0$.

We'll now show that $L$ is **linear**: Let $\phi$ and $\psi$ be functions.

$L[\phi + \psi] = (\phi + \psi)'' + p(\phi + \psi)' + q(\phi + \psi)$

$= \phi'' + \psi'' + p(\phi' + \psi') + q(\phi + \psi)$

$= \phi'' + p\phi' + q\phi + \psi'' + p\psi' + q\psi$

$= L[\phi] + L[\psi]$

For $c$ constant,

$L[c\phi] = (c\phi)'' + p(c\phi)' + q(c\phi)$

$= c\phi'' + cp\phi' + cq\phi = c(\phi'' + p\phi' + q\phi)$

$= cL[\phi]$

(This works because the differential equation is linear)

**Thm 3.2.2 (Principle of Superposition):** If $y_1$ and $y_2$ are solutions to the linear equation $y'' + p(t)y' + q(t)y = 0$, then $c_1y_1 + c_2y_2$ is a solution for any constants $c_1, c_2$.

**Pf:** Use the linear operator $L[\phi] = \phi'' + p\phi' + q\phi$. Then

$L[c_1y_1 + c_2y_2] = L[c_1y_1] + L[c_2y_2] = c_1L[y_1] + c_2L[y_2] = c_1y_1(0) + c_2y_2(0) = 0$

**Ex.1:** Show that $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are solutions to $y'' - y = 0$. Are $y_3(t) = \sinh(t)$ and $y_4(t) = \cosh(t)$ also solutions?

$y_1' = e^t$, $y_1'' = e^t$ so $(e^t) - (e^t) = 0 \ \checkmark$

$y_2' = -e^{-t}$, $y_2'' = e^{-t}$ so $(e^{-t}) - (e^{-t}) = 0 \ \checkmark$

By the principle of superposition, $c_1e^t + c_2e^{-t}$ is a solution for any $c_1$ and $c_2$.

$\sinh(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} \quad (c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}) \quad \text{So yes, these are solutions!}$

$\cosh(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} \quad (c_1 = \frac{1}{2}, c_2 = \frac{1}{2})$
So we know that if two solutions exist, then any linear combination is also a solution. Can we establish that solutions exist? Yes!

**Thm 3.2.1 (Existence and Uniqueness Thm):** Consider the IVP

\[ y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0. \]

If \( p, q, \) and \( g \) are continuous on an interval \( I, \) then there exists a unique twice-differentiable function \( y = \phi(t) \) satisfying the IVP.

**Ex 2:** On what interval is there guaranteed to be a unique twice-differentiable solution to \((t+1)y'' + u_n(1-t)y' + y = \sqrt{t}, \ y(\frac{1}{2}) = 1, y'(\frac{1}{2}) = 2?\)

\[ y'' + \frac{\ln(1-t)}{t+1} + \frac{y}{t+1} = \frac{\sqrt{t}}{t+1} \]

Discontinuous when \( t+1 = 0 \) \( \implies t = -1 \)

When \( 1-t \leq 0 \) \( \implies t \geq 1 \)

So continuous on \( 0 \leq t < 1. \)

We've addressed existence of solutions and how to form solutions from others. But how do we know we've found all solutions? How do we know, for example, that every solution to \( y'' - y = 0 \) can be written as \( c_1 e^{t} + c_2 e^{-t} \) for some constants \( c_1 \) and \( c_2? \)

To do this, we introduce the Wronskian. Given differentiable functions \( f \) and \( g, \) the Wronskian of \( f \) and \( g \) is the determinant

\[ \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g \]

**Thm 3.2.4:** Suppose that \( y_1 \) and \( y_2 \) are solutions to \( L[y] = y'' + p(t)y' + q(t)y = 0. \) Then the family of solutions \( y = c_1 y_1 + c_2 y_2 \) includes every solution if and only if the Wronskian of \( y_1 \) and \( y_2 \) is not zero for some point \( t_0. \)

(The proof comes from linear algebra - the solutions of \( L[y] = 0 \) are a vector space of dimension 2, and the Wronskian not being zero means that \( \{f, g\} \) is linearly independent, so it's a basis, so it spans the solution set. If a set of solutions \( \{y_1, y_2\} \) has Wronskian not equal to 0, we call it a fundamental set of solutions since \( c_1 y_1 + c_2 y_2 \) is the general solution, containing every solution. Fundamental sets are not necessarily unique.)
EX 3: Show that \( \{e^t, e^{-t}\} \) and \( \{\sinh(t), \cosh(t)\} \) are both fundamental sets of solutions to \( y'' - y = 0 \).

We've already seen that these are solutions, so we just need to check the Wronskian:

\[
\begin{vmatrix}
    e^t & e^{-t} \\
    e^t & e^{-t}
\end{vmatrix} = -2 \\
\begin{vmatrix}
    \sinh(t) & \cosh(t) \\
    \cosh(t) & \sinh(t)
\end{vmatrix} = -1
\]

\(-2 \neq 0\), so \( \{e^t, e^{-t}\} \) is a fundamental set.

\(-1 \neq 0\), so \( \{\sinh(t), \cosh(t)\} \) is a fundamental set.

EX 4: Show that \( \{e^{-t}, 2e^{-t}\} \) is not a fundamental set of solutions to \( y'' - y = 0 \).

\[
\begin{vmatrix}
    e^{-t} & 2e^{-t} \\
    -e^{-t} & -2e^{-t}
\end{vmatrix} = -2e^{-2t} - 2e^{-2t} = 0 \Rightarrow \text{not a fundamental set of solutions}
\]

EX 5: Suppose \( y_1 = x \) and \( y_2 = xe^x \) are solutions to \( L[y] = 0 \). Is \( \{y_1, y_2\} \) a fundamental set?

\[
\begin{vmatrix}
    x & xe^x \\
    1 & xe^x + e^x
\end{vmatrix} = x^2e^x + xe^x - xe^x = x^2e^x
\]

The Wronskian is not zero for some value of \( x \) (e.g., \( x = 1 \)), so it is a fundamental set.

EX 6: If the Wronskian \( W \) of \( f \) and \( g \) is \( 3e^{4t} \) and if \( f(t) = e^{2t} \), find \( g(t) \).

\[
\begin{vmatrix}
    e^{2t} & g \\
    2e^{2t} & g'
\end{vmatrix} = e^{2t}g' - 2e^{2t}g = 3e^{4t} \quad \text{(This is a first order ODE)}
\]

\[
g' - 2g = 3e^{2t}
\]

\[
\mu(t) = e^{-2t} \quad \Rightarrow \quad e^{-2t}g = \int e^{-2t} \cdot 3e^{2t} \, dt = \int 3 \, dt = 3t + C
\]

\[
g = 3 + e^{2t} + Ce^{2t} \quad \text{for some constant} \ C.\]