In Lesson 14, we dealt with mass-spring systems in which only gravity, the spring force, and a damping force acted on the mass. In this lesson, we assume an additional force is acting on the mass, given by the equation $g(t)$. Such a function is called a 

**Forcing function.**

Again using $F = ma$, we get

$$mg - 3u' + k(L + u') + g(t) = mu''$$

Gravity, damping force, spring force, additional force.

Recalling that $mg - KL = 0$, we end up with

$$mu'' + 3u' + ku = g(t)$$

(i.e. a non-homogeneous equation.) Generally, we'll focus on when $g(t)$ is a periodic function ($g(t) = \cos(at)$ or $\sin(at)$). So, the method of Undetermined Coefficients will be extremely useful.

When doing these problems, **ALWAYS** be very careful about units! Specify what your function represents.

**EX:** A mass of 2kg stretches a spring 8cm. The mass is acted upon by an external force of $3\cos(2t)$ N and has a damping force of magnitude 3N when the mass has a speed of 2cm/s. If the mass is initially at rest in its equilibrium position, set up an IVP for the position of the mass at time $t$.

$$F(t) = 3\cos(2t)$$

$$m = 2\text{kg}, \quad L = 0.08\text{m}, \quad \text{speed} = 0.02\text{m/s}$$

$$mg - KL = 0 \Rightarrow (2\text{kg})(9.8 \text{m/s}^2) - k(0.08\text{m}) = 0 \Rightarrow k = 245 \text{ kg/m}$$

Damping force is $3u'$, where $u'$ is velocity. If the damping force is 3N when speed is 0.02m/s, then $3(0.02) = 3 \Rightarrow 3 = 150 \text{ N/m}$.

So, $2u'' + 150u' + 245u = 3\cos(2t), \quad u(0) = 0, \quad u'(0) = 0$. (u in m, t in s)

If a periodic external force is applied to a vibrating system, there's an important concept called **resonance**. If the frequency $\omega$ of the external force is equal to the natural frequency $\omega_n$ of the vibrating system, you have resonance. Resonance has a huge impact on the system even when the external force is relatively small.

For example, in a seismograph, resonance can allow us to detect minor earthquakes which we could not feel ourselves. Resonance is also used as a way to tune instruments. If two instruments have similar (but not equal) frequency, a "beat" in the sound develops. Resonance occurs when in-tune.
If a bridge or building has a natural frequency which is similar to the frequency of external force, it can cause the collapse and destruction of the structure.

**Ex. 2:** A mass weighing 3 lb stretches a spring 1 in. Assume there is no damping on the mass. An external force of $2\cos(\omega t)$ lb is applied. Determine the value for $\omega$ for which resonance occurs.

**First, unit of force is lb. Standard unit for distance is ft.**

\[ mg = 3 \text{ lb} \times g = 32 \text{ ft/s}^2 \quad \text{so} \quad m = \frac{3}{32} \frac{\text{lb} \cdot \text{s}^2}{\text{ft}} = \frac{\frac{3}{32}}{\text{slug}} \quad L = 1 \text{ in} = \frac{1}{12} \text{ ft} \]

\[ mg - kL = 0 \implies 3 - \frac{3}{12} = 0 \implies k = 36 \text{ lb/ft} \]

\[ \frac{3}{32} u'' + 3\omega u = 2 \cos(\omega t) \]

Natural frequency is found through the complementary solution.

\[ \frac{3}{32} r^2 + 3\omega = 0 \]

\[ r^2 = -\frac{384}{3} \implies r = \pm \frac{i}{2} \sqrt{384} = \pm 8\sqrt{6} \quad i \]

\[ y_c(t) = C_1 \cos(8\sqrt{6}t) + C_2 \sin(8\sqrt{6}t) = R \cos(8\sqrt{6}t - \phi) \]

So resonance occurs when external force has frequency $\omega = 8\sqrt{6}$. 