Consider a differential equation of the form \( \frac{dy}{dt} = ay - b \) for \( a \) and \( b \) constants. We can solve this using clever techniques.

Notice:
\[
\frac{dy}{dt} = a \left( y - \frac{b}{a} \right) 
\]

So
\[
\frac{dy}{y - \frac{b}{a}} = a \, dt
\]

Integrating both sides yields:
\[
\ln \left| y - \frac{b}{a} \right| = at + C
\]

Exponentiate:
\[
\left| y - \frac{b}{a} \right| = e^{at+C} = e^{at} \cdot e^C
\]

This is just a constant.
\[
\left| y - \frac{b}{a} \right| = Ce^{at}
\]
\[
y - \frac{b}{a} = \frac{\pm}{e^{at}} \]
\[
y = \frac{b}{a} \pm Ce^{at}
\]

Again just a constant.
\[
y = Ce^{at} + \frac{b}{a}
\]

\( C \) is an arbitrary constant which results from indefinite integration. Whatever you plug in for \( C \) will give a valid solution here.

**Ex 1:** Solve \( \frac{dy}{dx} = 2y - 3 \)

\[
\frac{dy}{dx} = 2 \left( y - \frac{3}{2} \right)
\]
\[
\frac{dy}{y - \frac{3}{2}} = 2 \, dx
\]

\[
\ln \left| y - \frac{3}{2} \right| = 2x + C
\]
\[
y - \frac{3}{2} = \pm e^{2x} \cdot e^C
\]
\[
y - \frac{3}{2} = \pm Ce^{2x} \Rightarrow y = Ce^{2x} + \frac{3}{2}
\]

(Alternatively you could use the formula above.)

If we keep \( C \) as an arbitrary constant, \( y \) is said to be the general solution. If we choose a value for \( C \), \( y \) is called an integral curve.

Sometimes we are looking for a particular solution. In order to find it, we are often given an initial condition \( y(0) = y_0 \).

A differential equation with an initial condition is called an initial value problem (IVP).
**Ex 2:** Solve the IVP \( \frac{dy}{dt} = 3y - \omega, \ y(0) = 7 \)

**General Solution:** \( y = Ce^{3t} + 2 \)

Since \( y(0) = 7 \), \( 7 = Ce^{3\cdot0} + 2 = C + 2 \) \( \Rightarrow \ C = 5 \)

Thus the particular solution is \( y = 5e^{3t} + 2 \).

**Ex 3:** Verify that \( y(t) = 5e^{3t} + 2 \) is a solution to \( \frac{dy}{dt} = 3y - \omega \)

Note \( \frac{dy}{dt} = 15e^{3t} \)

New \( 3y - \omega = 3(5e^{3t} + 2) - \omega = 15e^{3t} + 6 - \omega = 15e^{3t} = \frac{dy}{dt} \)

Thus \( y = 5e^{3t} + 2 \) is indeed a solution.

**Ex 4:** Verify that \( y = e^{2t} \) is a solution to \( y'' - 2y' + 2y' - 4y = 0 \).

\( y' = 2e^{2t} \) \( \quad \) \( y'' = 4e^{2t} \) \( \quad \) \( y'' = 8e^{2t} \)

\( \Rightarrow 8e^{2t} - 2(4e^{2t}) + 2(2e^{2t}) - 4(e^{2t}) = 8e^{2t} - 8e^{2t} + 4e^{2t} + 4e^{2t} = 0 \)

**Ex 5:** Suppose a bacteria population grows at a rate proportional to its current population (i.e. \( \frac{dp}{dt} = rp \) where \( P(t) \) is population and \( r \) is the rate) If the population triples in 2 hours, what is \( r \)?

\( \frac{dp}{dt} = rp \)

\( \frac{dp}{p} = rdt \Rightarrow \ln|p| = rt + C \)

\( |p| = Ce^{rt} \Rightarrow P = Ce^{rt} \)

\( P(0) = C \) so \( P(2) = 3C \) and

\( 3C = Ce^{2r} \)

\( 3 = e^{2r} \)

\( \ln(3) = 2r \)

\( \frac{\ln(3)}{2} = r \)

**Ex 6:** For what values of \( r \) is \( y = e^{rt} \) a solution to \( y'' - y' - \omega y = 0 \)

\( (r^2 e^{rt}) - (re^{rt}) - \omega (e^{rt}) = 0 \)

\( = e^{rt}(r^2 - r - \omega) = e^{rt}(r - 3)(r + 2) = 0 \)

\( r = -2, 3 \)

If a differential equation has partial derivatives then it is a partial differential equation. If it only has ordinary derivatives, then it is an ordinary differential equation.

For example,

\( \frac{d^2y}{dt^2} + t \frac{dy}{dt} + y = 0 \) is ordinary

\( \frac{d^2y}{dt^2} + \frac{d^2y}{dx^2} = 0 \) is partial.
The order of a differential equation is the order of the highest derivative that appears in the equation. For example, the orders of the following are:

\[ \frac{d^2 y}{dt^2} + \frac{dy}{dt} + y = 0 \quad \text{is order 2} \]

\[ \frac{d^4 y}{dt^4} + y = 0 \quad \text{is order 4} \]

\[ \left( \frac{d^3 y}{dt^3} \right)^2 + y \frac{dy}{dt} = 1 \quad \text{is order 3} \]

An equation in \( n \) variables \( x_1, \ldots, x_n \) is **linear** if it's in the form

\[ a_1 x_1 + \cdots + a_n x_n = b \quad \text{where } a_1, \ldots, a_n, b \text{ are not variables.} \]

In a differential equation, we count \( y \) and its derivatives \( (y', y'', \ldots y^{(n)}) \) as the variables. A differential equation is **linear** if its of the form

\[ g_n(t) y^{(n)} + \cdots + g_1(t) y' + g_0(t) y = f(t) \quad \text{where } g_n(t), \ldots, g_0(t), f(t) \text{ are functions of the independent variable } t. \]

**Ex 7:** \( t^2 \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} = \cos(t) \) is linear

\[ e^t \frac{d^2 y}{dt^2} + \cos(t) \frac{dy}{dt} + y^2 = 1 \quad \text{is nonlinear (since } y \text{ is squared)} \]

\[ y \frac{dy}{dt} + 3 = 0 \quad \text{is nonlinear since } y \text{ is multiplied by } \frac{dy}{dt} \]

Classifying differential equations is important since some techniques of solving differential equations only apply to certain classes of differential equations. Lesson 3 covers the integrating factor method, which applies to first-order linear ODES.