Suppose we have a mass attached to a spring which is attached to another mass attached to a spring.

This situation is more complicated than with 1 mass. Suppose you have 2 interconnected tanks with salt water flowing into both and out of both.

In both of these examples, the behavior of object 1 is affected by the behavior of object 2 and vice versa. To handle these, we use systems of differential equations.

Ex 1: Salt water with concentration 3% of salt flows into tank 2 at a rate of 4 L/min. The well-stirred mixture from tank 2 flows into tank 1 at a rate of 4 L/min, and the well-stirred mixture of tank 1 flows out at a rate of 4 L/min. If tank 1 has 20 L, tank 2 has 30 L, tank 1 initially contains fresh water, and tank 2 initially has 6 g of salt, write a system of equations representing this situation.

Let $x_1(t)$ and $x_2(t)$ be the amount of salt in tanks 1 and 2 respectively at time $t$ minutes.

$$x_2' = \text{amount in} - \text{amount out}$$

$$= \frac{3g \cdot 4L}{L \text{ min}} - \frac{x_2(t) g \cdot 4L}{30L \text{ min}} = 12 - \frac{2}{15} x_2, \quad x_2(0) = 6.$$

$$x_1' = \text{amount in} - \text{amount out}$$

$$= \frac{x_2(t) g \cdot 4L}{30L \text{ min}} - \frac{x_1(t) g \cdot 4L}{20L \text{ min}} = \frac{2}{15} x_2 - \frac{1}{8} x_1, \quad x_1(0) = 0.$$
So our system is:

\[
\begin{align*}
X_1' &= -\frac{1}{8} X_1 + \frac{2}{15} X_2, \quad X_1(0) = 0 \\
X_2' &= -\frac{2}{15} X_2 + 12, \quad X_2(0) = 0
\end{align*}
\]

We can convert 2nd order equations into 1st order systems and vice versa.

**Ex2:** Convert \(3u'' + 2u' - u = \sin t, \quad u(0) = 3, u'(0) = 2\) into a system of first order equations.

Let \(X_1 = u\) and \(X_2 = u'\). Then \(X_1' = X_2\).

\[
\begin{align*}
3X_2' + 2X_2 - X_1 &= \sin t, \quad X_1(0) = 3 \\
3X_2' &= X_1 - 2X_2 + \sin t, \quad X_2(0) = u'(0) = 2
\end{align*}
\]

This gives the system of equations:

\[
\begin{align*}
X_1' &= X_2, \quad X_1(0) = 3 \\
X_2' &= \frac{2}{3} X_1 - \frac{2}{3} X_2 + \frac{1}{3} \sin t, \quad X_2(0) = 2
\end{align*}
\]

**Ex3:** Convert the system into a 2nd order equation.

\[
\begin{align*}
X_1' &= 3X_1 - 2X_2, \quad X_1(0) = 3 \\
X_2' &= 2X_1 - 2X_2, \quad X_2(0) = \frac{1}{2}
\end{align*}
\]

First, take the derivative of the second equation:

\[
X_2'' = 2X_1' - 2X_2'
\]

Substitute the first equation into \(X_1'\):

\[
X_2'' = 2(3X_1 - 2X_2) - 2X_2
\]

\[
X_2'' = 6X_1 - 4X_2 - 2X_2
\]

Solve the second equation for \(X_1\):

\[
2X_1 = X_2' + 2X_2
\]

Plug in:

\[
X_2'' = 6 \left( \frac{X_2' + X_2}{2} \right) - 4X_2 - 2X_2
\]

\[
X_2'' = 3X_2' + 6X_2 - 4X_2 - 2X_2 = 2X_2 + X_2'
\]

\[
X_2'' - X_2' - 2X_2 = 0
\]

We have \(X_2(0) = \frac{1}{2}\), so we just need \(X_2'(0)\). By the second equation,

\[
X_2'(0) = 2X_1(0) - 2X_2(0) = 2(3) - 2(\frac{1}{2}) = 5
\]

So our 2nd order equation is: \(X_2'' - X_2' - 2X_2 = 0, \quad X_2(0) = \frac{1}{2}, \quad X_2'(0) = 5\).
We will often work with systems of equations in matrix form. An $m \times n$ matrix $A = (a_{ij})$ is an array with $m$ rows and $n$ columns.

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{m1} \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix}$$

where $a_{ij}$ is the entry in row $i$, column $j$. In this class, we restrict our attention to $2 \times 2$ matrices and $2$-vectors. An $n$-dimensional row vector is a $1 \times n$ matrix and an $m$-dimensional column vector is an $m \times 1$ matrix.

If $A$ and $B$ are both $m \times n$ matrices, $A + B$ is defined by component-wise addition, i.e. if $(c_{ij}) = A + B$, then $c_{ij} = a_{ij} + b_{ij}$.

If $A$ is an $m \times n$ matrix and $c$ is a constant, then $cA = (ca_{ij})$ i.e. each component of $A$ is multiplied by $c$.

Multiplying two matrices is tricky. $AB$ only exists if $A$ has as many columns as $B$ has rows.

If $(c_{ij}) = AB$, the $c_{ij}$ is the dot product of the $i$th row of $A$ with the $j$th column of $B$.

In general $AB \neq BA$.

**Ex 4:** Let $A = \begin{pmatrix} 2+i & -i \\ 3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} i & 1 \\ 1 & 2 \end{pmatrix}$.

**Compute**

(a) $2A - B$

$$= 2 \begin{pmatrix} 2+i & -i \\ 3 & 0 \end{pmatrix} - \begin{pmatrix} i & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+2i & -2i \\ 6 & 0 \end{pmatrix} + \begin{pmatrix} -i & -1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 4+i & -2i-1 \\ 5 & -2 \end{pmatrix}$$

(b) $AB$

$$\begin{pmatrix} 2+i & -i \\ 3 & 0 \end{pmatrix} \begin{pmatrix} i & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2i-1-i & 2+i-2i \\ 3i+0 & 3+0 \end{pmatrix} = \begin{pmatrix} -1+i & 2-i \\ 3i & 3 \end{pmatrix}$$
(c) \[
\begin{bmatrix}
1 & 1 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
2 + i & -i \\
3 & 0
\end{bmatrix} = \begin{bmatrix}
2i - 1 + 3 & 1 - 0 \\
2 + i + 0 & -i + 0
\end{bmatrix} = \begin{bmatrix}
2i + 2 & 1 \\
8 + i & -i
\end{bmatrix}
\]

We can write systems of differential equations in terms of matrices
\[
\begin{align*}
x_1' &= 2x_1 + 3x_2 \\
x_2' &= 4x_1 - 2x_2
\end{align*}
\]

or simply
\[
\begin{bmatrix}
x_1' \\
x_2'
\end{bmatrix} = \begin{bmatrix}
2 & 3 \\
4 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

EX5: Verify that \(
\begin{bmatrix}
1 \\
-2
\end{bmatrix}e^{-4t}
\)
is a solution to the above system.

\[
\begin{bmatrix}
1 \\
-2
\end{bmatrix}e^{-4t} = \frac{1}{8}e^{-4t}
\]

\[
\begin{bmatrix}
2 & 3 \\
4 & -2
\end{bmatrix}
\begin{bmatrix}
1 \\
-2
\end{bmatrix}e^{-4t} = \begin{bmatrix}
2 - 6 \\
4 + 4
\end{bmatrix}e^{-4t} = \begin{bmatrix}
-4 \\
8
\end{bmatrix}e^{-4t}
\]

If you want, you can rewrite this as
\[
x_1(t) = e^{-4t} \quad x_2(t) = -2e^{-4t}
\]