Lesson 3: Integrating Factors

In this lesson, we focus on first order linear ODEs. Recall that such an ODE is of the form \( \frac{dy}{dt} + p(t)y = g(t) \)
(If \( \frac{dy}{dt} \) has a coefficient function, we can divide by it to put it in this form.)

Gottfried Leibniz noticed that the LHS of this equation resembles the result of the product rule on some function \( \mu(t)y \)
Recall \( \frac{d}{dt}[\mu(t)y] = \mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \frac{d}{dt}[\mu(t)y] \)

Such \( \mu(t) \) is called the integrating factor.

So we want
\[ \mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y \]

Subtracting \( \mu(t)\frac{dy}{dt} \) from both sides and dividing by \( y \) gives the differential equation
\[ \frac{d\mu(t)}{dt} = p(t)\mu(t) \]

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So \( \mu(t) = e^{\int p(t)dt} \) and \( \mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \frac{d}{dt}[\mu(t)y] \)

Since \( \frac{d}{dt} + p(t)y = g(t) \), \( \mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t) \). Note the LHS is equivalent to \( \mu(t)y \).

Integrating both sides (using FTC), \( \mu(t)y = \int \mu(t)g(t)dt + C \). We can then solve for \( y \) to get
\[ y(t) = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)g(s)ds + \frac{C}{\mu(t)} \]

for some convenient choice of \( t_0 \). Sometimes \( \int_{t_0}^{t} \mu(s)g(s)ds \) is not solvable using familiar functions, so we can leave the solution in this form. In that case, however, we will often deal with functions where the integral can be solved.

Method of Integrating Factors:
1) Given a first order linear ODE, write it in the form \( y' + p(t)y = g(t) \)
2) Compute the integrating factor \( \mu(t) = e^{\int p(t)dt} \)
3) Have the formula \( \frac{d}{dt}[\mu(t)y] = \mu(t)g(t) \).
4) Integrate both sides.
\[ \mu(t)y = \int \mu(t)g(t)dt + C \]
5) Solve for \( y \).
Ex 1: Solve the differential equation and describe the behavior of solutions as $t \to \infty$

$$ty' + y = 3t \cos 2t, \quad t > 0$$

$$y' + \frac{1}{t}y = 3\cos 2t$$

$$\mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln|t|} = 1 \quad \text{since} \ t > 0.$$  

$$\frac{d}{dt} [ty] = 3t \cos (2t), \quad \text{so} \quad ty = \int 3t \cos (2t) dt$$

Use integration by parts:  

$$u = 3t, \quad v = \frac{1}{2} \sin (2t)$$

$$du = 3 dt, \quad dv = \cos (2t) dt$$

$$ty = \frac{3}{2} t \sin (2t) - \int \frac{3}{2} \sin (2t) dt + C$$

$$ty = \frac{3}{2} t \sin (2t) + \frac{3}{4} \cos (2t) + C$$

$$y(t) = \frac{3}{2} \sin (2t) + \frac{3}{4} \cos (2t) + \frac{C}{t}$$

As $t \to \infty$, $\frac{3}{4} \cos (2t)$ and $\frac{C}{t}$ both go to 0. So as $t \to \infty$, $y$ is asymptotic to $\frac{3}{2} \sin (2t)$.

Ex 2: Consider $2y' - y = e^{-t/3}, \ y(0) = a$

(a) Draw a direction field and see if behavior depends on $a$.

--- use dfIELD8

$$y' = \frac{1}{2} y + \frac{1}{2} e^{t/3}, \quad \text{you'll see it does depend!}$$

(b) Solve the equation and find the value $a_0$ where the transition from one type of behavior to another occurs.

$$y' - \frac{1}{2} y = \frac{1}{2} e^{t/3}$$

$$\Rightarrow \mu(t) = e^{\int \frac{1}{2} dt} = e^{-\frac{1}{2} t}$$

$$e^{-\frac{1}{2} t} y = \mu(t) y = \int \mu(t) g(t) dt = \int \frac{1}{2} e^{-\frac{t}{6}} dt = -3 e^{-\frac{t}{6}} + C$$

$$y(t) = -3 e^{-\frac{t}{6}} + C e^{t/2}$$

$$a = y(0) = -3 + C \quad \text{so} \quad C = a + 3$$

$$y(t) = -3 e^{-\frac{t}{6}} + (a+3) e^{t/2}$$

$$a_0 = -3 \quad \text{since if} \ a_0 > -3, \ (a+3) e^{t/2} \text{is positive and dominates,}$$

$$\text{if} \ a_0 < -3, \ (a+3) e^{t/2} \text{is negative}.$$

Ex 3: Consider $y' + \frac{1}{2} y = 2 - t, \ y(0) = y_0$. Find the value of $y_0$ for which the solution touches, but does not cross, the $t$-axis.

$$\mu(t) = e^{\int \frac{1}{2} dt} = e^{\frac{1}{2} t}$$

$$\mu(t) y = e^{\frac{1}{2} t} y = \int \mu(t) g(t) dt = \int e^{\frac{1}{2} t} (2 - t) dt = \int 2 e^{\frac{1}{2} t} dt - \int t e^{\frac{1}{2} t} dt$$

$$e^{\frac{1}{2} t} y = 4 e^{t/2} - \int t e^{t/2} dt$$

$$u = t, \quad v = 2 e^{t/2} dt$$

$$du = dt, \quad dv = e^{t/2} dt$$
\[ e^{\frac{t}{2}} y = 4 e^{\frac{t}{2}} - (2 e^{\frac{t}{2}} - 4 e^{\frac{t}{2}}) + C = 8 e^{\frac{t}{2}} - 2 t e^{\frac{t}{2}} + C \]

\[ y = 8 - 2 t + C e^{-\frac{t}{2}} \]

\[ y_0 = y(\omega) = 8 + C \quad \Rightarrow \quad C = 8 + y_0 \]

\[ \Rightarrow y(t) = 8 - 2 t + (y_0 - 8) e^{-\frac{t}{2}} \]

In order to find where it touches but doesn’t cross the t-axis, must have a minimum or maximum touching the t-axis. Find the t-values where min or max occurs:

\[ y'(t) = -2 - \frac{1}{2} (y_0 - 8) e^{-\frac{t}{2}} \]

Set

\[ e^{-\frac{t}{2}} = \frac{-4}{y_0 - 8} \Rightarrow t = -2 \ln \left( \frac{-4}{y_0 - 8} \right) \]

Need \( y(t) = 0 \) at this t-value.

\[ 0 \Rightarrow 8 - 2 \left( -2 \ln \left( \frac{-4}{y_0 - 8} \right) \right) + (y_0 - 8) e^{-\frac{t}{2}} \]

Use Wolfram Alpha to find \( y_0 \approx -2.8731 \) or \( 8 - 4 e \).

(For a sanity check, graph \( y(t) = 8 - 2 t + (8 - 4 e - 8) e^{-\frac{t}{2}} \)).

**Ex:4**: Show that all solutions of \( 4 y' + t y = 4 \) approach a limit as \( t \to \infty \) and find the limiting value.

\[ y' + \frac{1}{4} y = 1 \]

\[ u(t) = e^{\frac{t}{4}} dt = e^{\frac{t^2}{8}} \]

\[ e^{\frac{t^2}{8}} y = \int e^{\frac{t^2}{8}} dt + C \]

\[ y = \frac{\int_{\infty}^{t} e^{\frac{t^2}{8}} dt}{e^{\frac{t^2}{8}}} + \frac{C}{e^{\frac{t^2}{8}}} \]

As \( t \to \infty \), \( \int_{\infty}^{t} \to \infty \) (indeterminate)

Apply L’Hospital’s Rule to the first term:

By FTC, \( \frac{d}{dt} \int_{t_0}^{t} e^{\frac{t^2}{8}} dt = e^{\frac{t^2}{8}} \)

So \( \lim_{t \to \infty} \int_{t_0}^{t} e^{\frac{t^2}{8}} dt = \lim_{t \to \infty} e^{\frac{t^2}{8}} = \frac{4}{t} \cdot e^{\frac{t^2}{8}} = \lim_{t \to \infty} \frac{4}{t} = 0 \)

So \( \lim_{t \to \infty} y(t) = 0 \) for all solutions \( y(t) \).