1. Consider a tank which has 400 gallons of a salt-water mixture. Initially, the tank has 15 lbs of salt in it. Water flows into the tank at a rate of 30 gallons per minute, and there is 1/2 lb of salt per gallon. There is a mixing device in the tank which keeps the salt evenly distributed throughout the salt-water mixture. The salt-water mixture flows out of the tank at a rate of 30 gallons per minute. Find the concentration (lbs per gallon) of salt in the tank in the long run.

Let \( q(t) \) be the amount of salt (in lbs) in the tank after \( t \) min.

Let \( c(t) \) be the concentration of salt (in \( \text{lbs/gal} \)) after \( t \) min.

Let \( v(t) \) be the amount of liquid (in gal) in the tank after \( t \) min.

\[
\frac{dq}{dt} = \text{rate of salt in} - \text{rate of salt out}
\]

rate of salt in: \( \frac{\frac{1}{2} \text{ lb}}{\text{gal}} \cdot \frac{30 \text{ gal}}{\text{min}} = 15 \text{ lb/min} \)

rate of salt out: \( \frac{\frac{q(t)}{v(t)} \text{ lb}}{\text{gal}} \cdot \frac{30 \text{ gal}}{\text{min}} = \frac{30q}{400} \text{ lb/min} \) \( (v(t)=400) \)

\[
\frac{dq}{dt} = 15 - \frac{3q}{40} \quad \text{(can use integrating factor or separable)}
\]

Get \( q(t) = Ce^{-\frac{3t}{40}} + 200 \). Now \( q(0) = 15 \), so \( 15 = C + 200 \)

\( \Rightarrow C = -185 \) and \( q(t) = -185e^{-\frac{3t}{40}} + 200 \).

Notice \( c(t) = \frac{q(t)}{v(t)} = \frac{-185e^{-\frac{3t}{40}} + 200}{400} = -\frac{37}{80}e^{-\frac{3t}{40}} + \frac{1}{2} \)

\[
\lim_{t \to \infty} c(t) = \lim_{t \to \infty} \left( -\frac{37}{80}e^{-\frac{3t}{40}} + \frac{1}{2} \right) = \frac{1}{2}
\]

\( \Rightarrow \frac{1}{2} \text{ lb/gal} \)
2. Consider a tank which has 400 gallons of pure water, and has a capacity of 700 gallons. Salt water begins to flow into the tank at a rate of 5 gallons per minute and there are 10 grams of salt per gallon. There is a mixing device in the tank which keeps the salt evenly distributed throughout the tank. The mixture in the tank flows out at a rate of 3 gallons per minute. How much salt will be in the tank the instant it begins to overflow?

Use same symbols q(t), v(t) as in problem 1.

**Notice**: v(t) is not constant.

\[
\frac{dv}{dt} = \text{(rate in)} - \text{(rate out)} = 5 \, \text{gal/min} - 3 \, \text{gal/min} = 2 \, \text{gal/min}
\]

Thus \( v(t) = 2t + C \). \( v(0) = 400 \Rightarrow v(t) = 2t + 400 \)

\[
\frac{dq}{dt} = \text{(rate in)} - \text{(rate out)} = \frac{10g}{gal} \cdot \frac{5gal}{min} - \frac{q(t)}{2t+400} \cdot \frac{3gal}{min}
\]

\[
\frac{dq}{dt} = 50 - \frac{3q}{2t+400}
\]

Use integrating factor to obtain

\[
q(t) = \frac{C}{(t+200)^{3/2}} + 20t + 4000
\]

\( q(0) = 0 \) so \( 0 = \frac{C}{200^{3/2}} + 4000 \Rightarrow C = -8000000\sqrt{2} \)

\[
q(t) = \frac{-8000000\sqrt{2}}{(t+200)^{3/2}} + 20t + 4000
\]

Tank overflows when \( v = 700 \)

\( 700 = 2t + 400 \Rightarrow t = 150 \)

\( q(150) = 5,272.1 \text{g} \)
3. Pete stands at the top of a 40 meter building and throws a hammer upward with a speed of 5 m/s. Suppose there is a force due to air resistance acting on the hammer in the opposite direction of velocity with a magnitude of $\frac{15}{22}$ m/s. Set up a differential equation to model this scenario (use $g = 9.8 \text{ m/s}^2$ as the magnitude of the acceleration due to gravity).

Let $v(t)$ be the velocity at $t$ sec. Let $m$ be the mass of the hammer. 

\[ F = ma \]

Going up:

- gravity
- air resistance

\[ a = \frac{dv}{dt}, \quad \text{so} \quad -mg - \frac{v}{22} = m \frac{dv}{dt} \quad \text{or} \quad \frac{dv}{dt} = -9.8 - \frac{v}{22m} \]

Notice if $v > 0$, then $-\frac{v}{22} < 0$. If $v < 0$, then $-\frac{v}{22} > 0$. 

Thus this equation works for both going up and going down.

Initial conditions: $v(0) = 5$, $h(0) = 40$

\[ \frac{dv}{dt} = -9.8 - \frac{v}{22m} \]
4. Suppose that the rate of change of a function $f$ is proportional to a function $g$. Write a differential equation which expresses this situation.

\[ f' = kg, \text{ } k \text{ a constant} \]

5. Newton's Law of Cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of tea obeys Newton's Law of Cooling. Assume the tea has a temperature of 190°$F$ when freshly poured, and 2 minutes later has cooled to 175°$F$ in a room at 72°$F$. Find a function for the temperature $T$ of the tea at time $t$.

Let $T =$ temp (in °F) of tea after $t$ minutes.

$T_s =$ surrounding temp

\[ \frac{dT}{dt} = k(T - T_s) = k(T - 72) \]

$T(t) = Ce^{kt} + 72$

$T(0) = 190 \quad T(2) = 175$

190 = $C + 72$ $\Rightarrow$ $C = 118$

175 = 118$e^{2k} + 72$ $\Rightarrow$ \[ \frac{103}{118} = e^{2k} \Rightarrow k = \frac{1}{2} \ln \left( \frac{103}{118} \right) \]

$T(t) = 118e^{\ln\left(\frac{103}{118}\right)t} + 72 = 118\left(\frac{103}{118}\right)^t + 72$
6. Suppose that a rocket is launched straight up from the surface of the Earth with an initial velocity of \( v_0 = \sqrt{2gR} \), where \( R \) is the radius of the Earth. Neglect air resistance. Find an expression for the velocity \( v \) in terms of the distance \( x \) from the surface of the Earth. Find the time required for the rocket to go \( 140,000,000 \) miles (the approximate distance from Earth to Mars). Assume that \( R = 4000 \) miles. Assume that the acceleration due to gravity \( g = 32 \text{ ft/s}^2 \) (There are 5280 feet in a mile.)

\[
\begin{align*}
F &= ma, \text{ so } m \frac{dv}{dt} = -k = \frac{-k}{(x+R)^2}, \quad w(0) = -mg \\
\frac{dv}{dt} &= \frac{-gR^2}{(x+R)^2} \\
\text{By the Chain Rule, } \frac{dv}{dt} &= \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx} \quad (\text{since } v = \frac{dx}{dt}) \\
v \frac{dv}{dx} &= \frac{-gR^2}{(x+R)^2} \\
v(0) &= \frac{\sqrt{2gR}}{R+R} \quad \text{Integrate to get } v \\
\text{Integrate to get } x \\
\Rightarrow \sqrt{R+x} \frac{dx}{dt} &= \frac{\sqrt{2gR}}{R+R} \\
\frac{2}{3} (R+x)^{3/2} &= \frac{\sqrt{2gR}}{R+R} t + C \\
X(0) &= 0 \quad \Rightarrow C = \frac{2}{3} R^{3/2} \quad \text{To solve for time, use } X = 140,000,000, \quad R = 4000 \\
t &= \frac{\frac{2}{3} (R+x)^{3/2} - C}{R \sqrt{2g}} \\
g &= \frac{32 \text{ ft}}{s^2} . \frac{1 \text{ mi}}{5280 \text{ ft}} \quad \text{(gives } \text{mi/s}^2) \\
gives t \text{ in seconds.} \\
Divide \text{ by } 3600 \text{ to convert to hours.} \\
\Rightarrow t = 696,400.57 \text{ hrs}
\end{align*}
\]
7. If Jack weighs 200 lbs, what is his mass?

Weight is force due to gravity, so \( 200 \text{ lbs} = mg \)

\[
200 = 32m \\
m = \frac{200}{32} = 6.25 \text{ slugs} \quad (1 \text{ slug} = \frac{1 \text{ lb} \cdot \text{s}^2}{\text{ft}})
\]