Consider the differential equation

\[(2xy^2 + 2y) + (2x^2y + 2x)y' = 0\]

Is this equation linear? No! (integrating factors doesn't work)
Is it separable? No!
But we can notice something very interesting. Consider the function
\[\psi(x,y) = x^2y^2 + 2xy\]
\[\frac{d\psi}{dx} = \psi_x = 2xy^2 + 2y \quad \frac{d\psi}{dy} = \psi_y = 2x^2y + 2x\]

Knowing this, we can rewrite the above as \(\frac{d\psi}{dx} + \frac{d\psi}{dy} \cdot \frac{dy}{dx} = 0\)

Since \(\psi(x,y)\) is a function of \(x\) and \(y\), the multivariate chain rule tells us that
\[\frac{d\psi}{dx} = \frac{d\psi}{dx} \cdot \frac{dx}{dx} + \frac{d\psi}{dy} \cdot \frac{dy}{dx}\]

So really our differential equation is of the form \(\frac{d\psi}{dx} = 0\).
So we get \(\psi(x,y) = C\), and thus \(x^2y^2 + 2xy = C\) implicitly is a solution.

A differential equation of the form \(M(x,y) + N(x,y) dy/dx = 0\) is called exact if there exists a function \(\psi(x,y)\) with \(\psi_x = M(x,y)\) and \(\psi_y = N(x,y)\).

The solutions of an exact equation can be given implicitly by \(\psi(x,y) = C\) for \(C\) an arbitrary constant.

**Thm 2.6.1**: If \(M, N, M_y, N_x\) are continuous in the rectangular region \(R\):
\(a < x < b, \gamma < y < \delta\), then \(M(x,y) + N(x,y) \frac{dy}{dx} = 0\) is exact on \(R\) (i.e. there exists a function \(\psi(x,y)\) with \(\psi_x = M\) and \(\psi_y = N\)) if and only if \(M_y = N_x\).

**Ex 1**: Determine whether the following differential equations are exact:

(a) \((e^x \sin y + 2xy) + (e^x \cos y + x^2) \frac{dy}{dx} = 0\)

\[M(x,y) = e^x \sin y + 2xy \quad N(x,y) = e^x \cos y + x^2\]

\(M_y = e^x \cos y + 2x \quad N_x = e^x \cos y + 2x\)

\(M = N_x, \text{ so by Thm 2.6.1 the equation is exact.}\)

(b) \((3x^2 + y) - (2y + x) y' = 0\)

Be careful here! \(M(x,y) = 3x^2 + y, N(x,y) = -2y - x\).

\(M_y = 1 \quad N_x = -1 \Rightarrow M_y \neq N_x, \text{ so not exact.}\)

(c) \(y' = \frac{(\omega y + 2x)}{3y^2 - \omega x} \Rightarrow (3y^2 - \omega x) y' = (\omega y + 2x)\)

\(-\omega y - 2x) + (3y^2 - \omega x) y' = 0\)

\(M = -\omega \quad N_x = -\omega \Rightarrow \text{equation is exact.}\)
So if we have an exact equation, how can we figure out what \( \psi(x,y) \) is?

We know \( \psi_x(x,y) = M(x,y) \) and \( \psi_y(x,y) = N(x,y) \). If we integrate \( M(x,y) \) with respect to \( x \) we should get \( \psi(x,y) \) (up to a function of \( y \)).

We can then differentiate our result with respect to \( y \) and that should equal \( N(x,y) \). By using another integration, we can find \( \psi(x,y) \) exactly.

**EX2:** Solve \( (e^x \sin y + 2xy) + (e^x \cos y + x^2) \frac{dy}{dx} \)

In **Ex1**, we checked that this is exact. Thus there exists a function \( \psi(x,y) \)

s.t. \( \psi_x(x,y) = M(x,y) = e^x \sin y + 2xy \) and \( \psi_y(x,y) = N(x,y) = e^x \cos y + x^2 \)

\[
\psi(x,y) = \int \psi_x(x,y) \, dx = \int e^x \sin y + 2xy \, dx = e^x \sin y + x^2 y + h(y)
\]

where \( h(y) \) is some function of \( y \):

\[
\psi_y(x,y) = \frac{\partial}{\partial y} [e^x \sin y + x^2 y + h(y)] = e^x \cos y + x^2 + h'(y)
\]

But \( \psi_y(x,y) = N(x,y) = e^x \cos y + x^2 \Rightarrow h'(y) = 0 \)

\( h(y) = \int h'(y) \, dy = \int 0 \, dy = 0 + C_1 \)

Thus \( \psi(x,y) = e^x \sin y + x^2 y + C_1 = e^x \sin y + x^2 y + C_1 \)

Hence, \( e^x \sin y + x^2 y + C_1 = C_2 \)

\( \Rightarrow e^x \sin y + x^2 y = C_2 - C_1 = C \) (another arbitrary constant)

And \( e^x \sin y + x^2 y = C \) is a solution.

**EX3:** Solve \( y' = \frac{e^y + 2x}{3y^2 - 6x} \)

We showed this was exact in **Ex1**, thus there exists a function \( \psi(x,y) \)

s.t. \( \psi_x = -e^y - 2x \) and \( \psi_y = 3y^2 - 6x \)

\[
\psi(x,y) = \int \psi_x \, dx = \int -e^y - 2x \, dx = -e^y x - x^2 + h(y)
\]

\[
\frac{d}{dy} \psi = -e^y x + h'(y) = \psi_y = 3y^2 - 6x \Rightarrow h'(y) = 3y^2
\]

\( h(y) = \int 3y^2 \, dy = y^3 + C \) (but the constant doesn't matter)

\[
\psi(x,y) = -e^y x^2 + y^3 \quad \text{and} \quad -e^y x^2 + y^3 = C \text{ is a solution.}
\]

**EX4:** Solve the IVP and determine where the solution is valid.

\[
(2x - y) + (2y - x) \frac{dy}{dx} = 0 \quad y(1) = 3
\]

\( M_y = -1 \quad N_x = -1 \Rightarrow \text{exact!} \)

\[
\psi(x,y) = \int 2x - y \, dx = x^2 - xy + h(y)
\]

\[
\frac{d}{dy} \psi = -x + h'(y) = 2y - x \Rightarrow h'(y) = 2y \Rightarrow h = y^2
\]
So \( x^2 - yx + y^2 = c \) is a solution. \( y(1) = 3 \Rightarrow 1^2 - 3(1) + 3^2 = c \Rightarrow c = 7 \)

\[
x^2 - yx + y^2 = 7 \Rightarrow x^2 - yx + y^2 - 7 = 0 \quad \text{or} \quad y^2 - xy + x^2 - 7 = 0
\]

**Quadratic Formula:**

\[
y = \frac{x \pm \sqrt{x^2 - 4(y^2 - 7)}}{2} = \frac{x \pm \sqrt{x^2 - 4x^2 + 28}}{2}
\]

\[
y = \frac{x \pm \sqrt{28 - 3x^2}}{2}
\]

In order to satisfy the initial condition

\[
3 = \frac{1 \pm \sqrt{28 - 3}}{2} = \frac{1 \pm 5}{2}
\]

so \( y = \frac{1}{2} \left( x + \sqrt{28 - 3x^2} \right) \). This is valid when \( 28 - 3x^2 \geq 0 \)

\[
28 \geq 3x^2
\]

\[
\frac{28}{3} \geq x^2
\]

\[
\sqrt{\frac{28}{3}} \geq 1 \times 1
\]

\[
-\sqrt{\frac{28}{3}} \leq x \leq \sqrt{\frac{28}{3}}
\]

If \( x = \frac{-1}{2} \sqrt{\frac{28}{3}} \), \( y = \frac{1}{2} \left( \pm \sqrt{\frac{28}{3}} \right) \)

Looking back at the original differential equation:

\[
\left( 2 \left( \pm \sqrt{\frac{28}{3}} \right) + \frac{1}{2} \sqrt{\frac{28}{3}} \right) + \left( 2 \left( \frac{-1}{2} \sqrt{\frac{28}{3}} \right) + \sqrt{\frac{28}{3}} \right) y' = 0
\]

\[
= 0
\]

So the differential equation is not satisfied. Thus the solution is valid when \( |x| < \sqrt{\frac{28}{3}} \) or equivalently when \( -\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}} \).