How (Information Theoretically) Optimal Are Distributed Decisions?

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***Abstract*—"If we know more, we can achieve more." This adage also applies to networks, where more information about the network state translates into higher sum-rates. In this paper, we formalize this increase of sum-rate with increased knowledge of network. The knowledge of network is measured in terms of the number of hops of information about the network while the sum-rate is normalized by the maximum sum-rate that can be achieved with complete information. As the knowledge about the network increase, the achievable normalized sum-rate also increases. The best normalized sum-rate is called normalized sum-capacity. In this paper, we characterize the increase of normalized sum-capacity with the hops of information about the network for many classes of deterministic interference networks for the cases of one and two-hops of instantaneous channel information.**

I. Introduction

The node mobility in wireless networks leads to constant changes in network connectivity at long time-scales and network state at short time-scales. The optimal rate allocation and associated encoding and decoding rules depend on both the network connectivity and the current network state. However, in large wireless networks, acquiring full network connectivity and state information for making optimal decisions is typically infeasible. As a result, nodes have to make distributed decisions about transmission and reception parameters. The key question then is how do distributed decisions perform when compared to the optimal decisions which have full network state information.

The question was first initiated in [1, 2], where the authors used a message-passing abstraction of network protocols to formulate a metric of limited network view at each node in the form of number of message rounds; each message round adds two extra hops of channel information at the transmitters. The key result was that distributed decisions can be either sum-rate optimal or can be arbitrarily worse than the global-information sum-capacity. This result was further strengthened for arbitrary K-user interference network in [3], where the authors characterized all network connectivities to allow optimal distributed rate allocation with two hops of network information at each transmitter. In this paper, we will focus on a deterministic approximation of Gaussian interference channels [4, 5].

In this paper, we take the next major step in understanding the performance of distributed decisions. We compute the capacity of distributed decisions for several network topologies with one-hop and two-hop network information at the transmitter. We show that in most cases, a maximal independent graph (MIG) scheduling algorithm achieves the maximum sum-rate among all distributed encoding and decoding schemes, when the transmitters have no more than two hops of channel information. The MIG schedule is shown to be optimal for most three-user bipartite interference topologies, K-user cyclic chain and K-user d-to-many interference channel. A maximal independent graph is defined as a sub-graph which admits a distributed encoding and decoding scheme which achieves same sum-capacity as a scheme with full global information.

For one hop information at the transmitters, maximal independent graphs are equivalent to maximal independent sets (MIS), which are largest subsets with non-interfering transmitter-receiver pairs. Note maximal independent set scheduling or maximal weighted independent sets are often the optimal schedules under traditional SINR based protocol models for networks. Our results show that for many topologies, MIS schedule is information-theoretically optimal. For two-hop information, MIS schedule is largely suboptimal compared to a general graph schedule, where interfering flows are scheduled simultaneously and advanced network-centric coding and decoding schemes are used (like superposition coding and successive decoding). Two-hop information about the network allows the nodes to adopt such policies and are only possible because of the extra network state information.

However, the MIG (for two-hop) and its special case, MIS (for one-hop), scheduling is not optimal in general for all network topologies. We show that in specific network topologies, higher rates are achievable by using a more sophisticated coding schemes. For the case of one-hop information in 3-user cyclic chain network, we show that a coded MIS (CMIS) schedule, where the coding is performed over two scheduling time-slots achieves a higher rate than pure scheduling based network. For the case of two-hop information, a weighted MIG schedule achieves the optimal distributed performance for some special network topologies.

The rest of the paper is organized as follows. In Section

II, we give some preliminaries and definitions that will be used throughout the paper. In Section III, we define maximal independent graph scheduling. In Section IV, we characterize the performance of maximal independent graph scheduling and generalize it to coded maximal independent set scheduling with one-hop information and provide an example topology when there exists a better algorithm for two hop information. Section V concludes the paper.

II. Channel Model and Problem Formulation

A. Channel Model

We consider the deterministic [4] interference channel with K transmitters and K receivers, where the input of the k^{th} transmitter at time i can be written as $X_k[i] = \left[X_{k_1}[i]\ X_{k_2}[i]\ \ldots X_{k_q}[i]\right]^T$, $k = 1,2,\cdots,K$, such that $X_{k_1}[i]$ and $X_{k_q}[i]$ are the most and the least significant bits, respectively. The received signal of user j, $j = 1,2,\cdots,K$, at time i is denoted by the vector $Y_j[i] = \left[Y_{j_1}[i]\ Y_{j_2}[i]\ \ldots\ Y_{j_q}[i]\right]^T$. Associated with each transmitter k and receiver j is a non-negative integer n_{kj} that represents the gain of the channel between them. The received signal $Y_j[i]$ is given by

$$Y_j[i] \quad = \quad \sum_{k=1}^{K} \mathbf{S}^{q-n_{kj}} X_k[i] \tag{1}$$

where q is the maximum of the channel gains (*i.e.* $q = \max_{j,k}(n_{jk})$), the summation is in $\mathbb{F}_2^q$, and $\mathbf{S}^{q-n_{jk}}$ is a $q \times q$ shift matrix with entries $\mathbf{S}_{m,n}$ that are non-zero only for $(m,n) = (q - n_{jk} + n, n), n = 1,2,\ldots,n_{jk}$.

The network can be represented by a square matrix H whose $(j,k)^{th}$ entry is n_{jk}. Note H need not be symmetric. We assume that that there is a direct link between every transmitter T_i and its intended receiver D_i. On the other hand, if a cross-link between transmitter i and receiver j does not exist, then $H_{ij} \equiv 0$. Given a network, its network is a set of edges $\mathsf{E} = \{(\mathsf{T}_i, \mathsf{D}_j)\}$ such that a link $\mathsf{T}_i - \mathsf{D}_j$ is not identically zero. Then the set of network states, $\mathcal{G}$, is the set of all weighted graphs defined on E. Note that the channel gain can be zero but not guaranteed[1] to be if the node pair $(\mathsf{T}_i, \mathsf{D}_j) \in \mathsf{E}$.

B. Normalized sum-capacity

For each user k, let message index m_k be uniformly distributed over $\{1,2,...,2^{nR_k}\}$. The message is encoded as X_k^n using the encoding functions $e_k(m_k|N_k, \mathsf{SI}) : \{1,2,\ldots,2^{nR_k}\} \mapsto \{0,1\}^{nq}$, which depend on the local view, N_k, and side information about the network, SI. The message is decoded at the receiver using the decoding function $d_k(Y_k^n|N_k', \mathsf{SI}) : \{0,1\}^{nq} \mapsto \{1,2,\ldots,2^{nR_k}\}$, where N_k' is the receiver local view and SI is the side information. The corresponding probability of decoding error $\lambda_k(n)$ is defined as $\Pr[m_k \neq d_k(Y_k^n|N_k', \mathsf{SI})]$. A rate tuple $(R_1, R_2, \cdots, R_K)$ is said to be achievable if there exists a sequence of codes such that the error probabilities $\lambda_1(n), \cdots \lambda_K(n)$ go to zero as n goes to infinity for all network states consistent with the side information. The sum-capacity is the supremum of $\sum_i R_i$ over all possible encoding and decoding functions.

We will use hop count (h) as a metric to account for the local information at the transmitters. We say that there is h-local information when all the transmitters knows only those links which are at-most h hop distant from them while the receivers know only those links which are at-most $h+1$ hop distant from them. The links incident on a node are 1-hop distant and the hop-distance of a link from a node is one plus the minimum amount of links to traverse starting from the node till the link.

A strategy is define as the set of encoding functions at all the transmitters. We will now define normalized sum-rate and normalized sum-capacity that will be used throughout the paper. These notions represent the percentage of the full-knowledge sum-capacity that can be achieved with partial information.

Definition 1. *Normalized sum-rate of α is said to be achievable in a set of network states with partial information if there exists a strategy that each of the transmitter i uses based on its local information N_i and side information SI, such that following holds. The strategy yields a sequence of codes having rates R_i at the transmitter i such that the error probabilities at the receiver, $\lambda_1(n), \cdots \lambda_K(n)$, go to zero as n goes to infinity, satisfying*

$$\sum_i R_i \geq \alpha C_{sum}$$

for all the sets of network states consistent with the side information. Here C_{sum} is the sum-capacity of the whole network with the full information.

Definition 2. *Normalized sum-capacity, α^*, is defined as the supremum over all achievable normalized sum rates α.*

Earlier, we defined a notion of *universal optimality* [2], which is now a special case of our current definition. A universally optimal strategy is simply the one which achieves $\alpha^*(h) = 1$ in the desired network. Hence, our main result in [3] characterizes *all* single-hop interference networks which has normalized sum-capacity equal to one ($\alpha^*(2) = 1$).

III. Maximal Independent Graph Scheduling

In this section, we will define a graph scheduling algorithm which can be used with h hops of information which we define as "Maximal Independent Graph Scheduling" (MIG Scheduling).

Let $A_1, A_2, \cdots, A_t$ be t sets, $A_i \subseteq \{1,2,\cdots,K\}$ for all $1 \leq i \leq t$, such that for each $1 \leq i \leq t$, if all links connected to users in A_i^c are disconnected, the equivalent connectivity satisfies $\alpha^*(h) = 1$. Let

$$d = \min_{j=1}^{K} \sum_{i=1}^{t} \mathbf{1}_{\{j\} \subseteq A_i}. \tag{2}$$

[1]The model is inspired by fading channels, where the existence of a link is based on its average channel gain. On the average the link gain may be above noise floor but its instantaneous value can be below noise floor.

Choose the sets that maximize d/t. MIG Scheduling algorithm uses t time-slots, with users in A_i^c turning off in i^{th} time-slot and the users in A_i use a strategy that achieves $\alpha^*(h) = 1$ with the connectivity containing of only users in A_i.

Theorem 1. *MIG Scheduling achieves a normalized sum-rate of d/t.*

We will first describe the conditions for $\alpha^*(h) = 1$ when $h = 1, 2$ and then give examples for this scheduling algorithm.

Theorem 2. *Normalized sum-capacity of a K-user interference channel with one-hop knowledge is equal to one (*i.e. $\alpha^*(1) = 1$*) if and only if all the receivers have degree* 1.

Theorem 3 ([3])**.** *Normalized sum-capacity of a K-user interference channel with two-hop knowledge is equal to one (*i.e. $\alpha^*(2) = 1$*) if and only if all the connected components of the topology are in one-to-many configuration or a fully-connected configuration.*

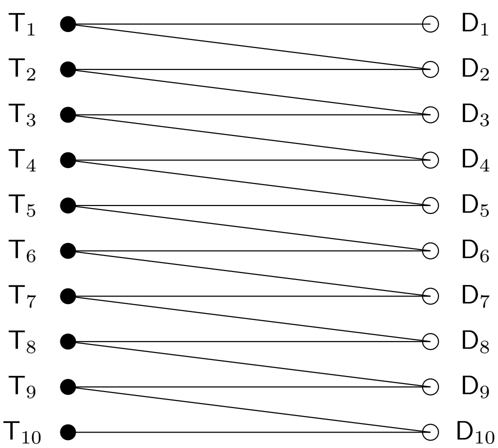

Fig. 1. Z-channel chain with 10 users.

Let us consider as an example, a Z-chain consisting of $K = 10$ users as shown in Figure 1. With one hop information at each transmitter, MIG Scheduling algorithm can be described as follows. Let $A_1 = \{1, 3, 5, 7, 9\}$ and $A_2 = \{2, 4, 6, 8, 10\}$. We use a two-time slotted strategy. In the first time-slot, odd users transmit while the even users remain silent while the reverse happens in the second time-slot. MIG Scheduling achieves $\alpha(1) = 1/2$.

With two-hop information at each transmitter, MIG Scheduling algorithm can be described as follows. Let $A_1 = \{1, 2, 4, 5, 7, 8, 10\}$, $A_2 = \{1, 3, 4, 6, 7, 9, 10\}$ and $A_3 = \{2, 3, 5, 6, 8, 9\}$. According to the MIG Scheduling algorithm, three time-slots are used and users in A_i uses a strategy that achieves $\alpha(2) = 1$ in i^{th} time-slot. MIG Scheduling strategy achieves $\alpha(2) = 2/3$.

It can also be shown that $\alpha^*(1) = 1/2$ and $\alpha^*(2) = 2/3$ thus proving that MIG Scheduling is optimal for this network.

IV. Performance of MIG Scheduling

We call the topology shown in figure 2 as a 3-user cyclic chain. MIG Scheduling is optimal for any 3-user interference channel except when the network is a 3-user cyclic chain. The following theorem gives many cases when MIG Scheduling is optimal

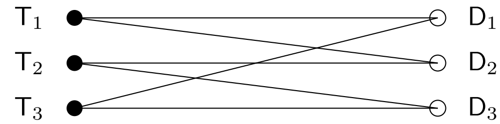

Fig. 2. 3-user cyclic chain.

Theorem 4. *MIG Scheduling is optimal for $h = 1$ when the network is of one of the following forms*

1) *Three user interference channel, except 3-user cyclic chain,*
2) *Z-channel chain, $\alpha^*(1) = 1/2$ for $K \geq 2$,*
3) *d-to-many interference channel, $\alpha^*(1) = \frac{1}{d+1}$ for $K \geq 2$ and $1 \leq d < K$,*
4) *many-to-d interference channel, $\alpha^*(1) = \frac{1}{d+1}$ for $K \geq 2$ and $1 \leq d < K$,*
5) *fully-connected interference channel, $\alpha^*(1) = \frac{1}{K}$.*

We will now understand the only case when MIG Scheduling is not optimal in a 3-user interference channel. The MIG Scheduling algorithm uses three time-slots scheduling user i in time-slot i. Thus, MIG Scheduling achieves $\alpha(1) = 1/3$. We will now describe another strategy for this example, which uses two time-slots as follows. In the first time-slot, we schedule $A_1 = \{1, 2\}$ and in second time-slot, we schedule $A_2 = \{2, 3\}$ such that the data of the second user is repeated in the two time-slots. Each user sends at the rate equal to the direct link capacity to the intended receiver. We will now show that the data can be decoded at the intended receivers. The first receiver can decode its data in the first time-slot since it receives no interference. The second receiver can similarly decode the data in the second time-slot. The third receiver on the other hand performs a xor operation of the data received in the two time-slots which gives a non-interfered direct signal which can be decoded. Thus, all the receivers can decode the data and this strategy achieves $\alpha(1) = 1/2$.

It is easy to see that for one-hop information, MIG Scheduling strategy reduces to Maximal Independent Set Scheduling (MIS Scheduling that can be described as follows. Consider a partition of $\{1, \cdots, K\}$ with each set containing mutually non-interfering users such that number of sets in the partition is minimum. Let there be t such sets, $A_1, \cdots, A_t$. Users in set A_i are scheduled in i^{th} time-slot. Since, each user is scheduled in only one out of t-time-slot, MIS Scheduling achieves $\alpha(1) = 1/t$.

This leads us to generalize MIS Scheduling strategy for 1 hop information. This strategy is labelled as Coded Maximal Independent Set Scheduling (CMIS Scheduling). CMIS Scheduling can be described as follows. Let $A_1, A_2, \cdots, A_t$

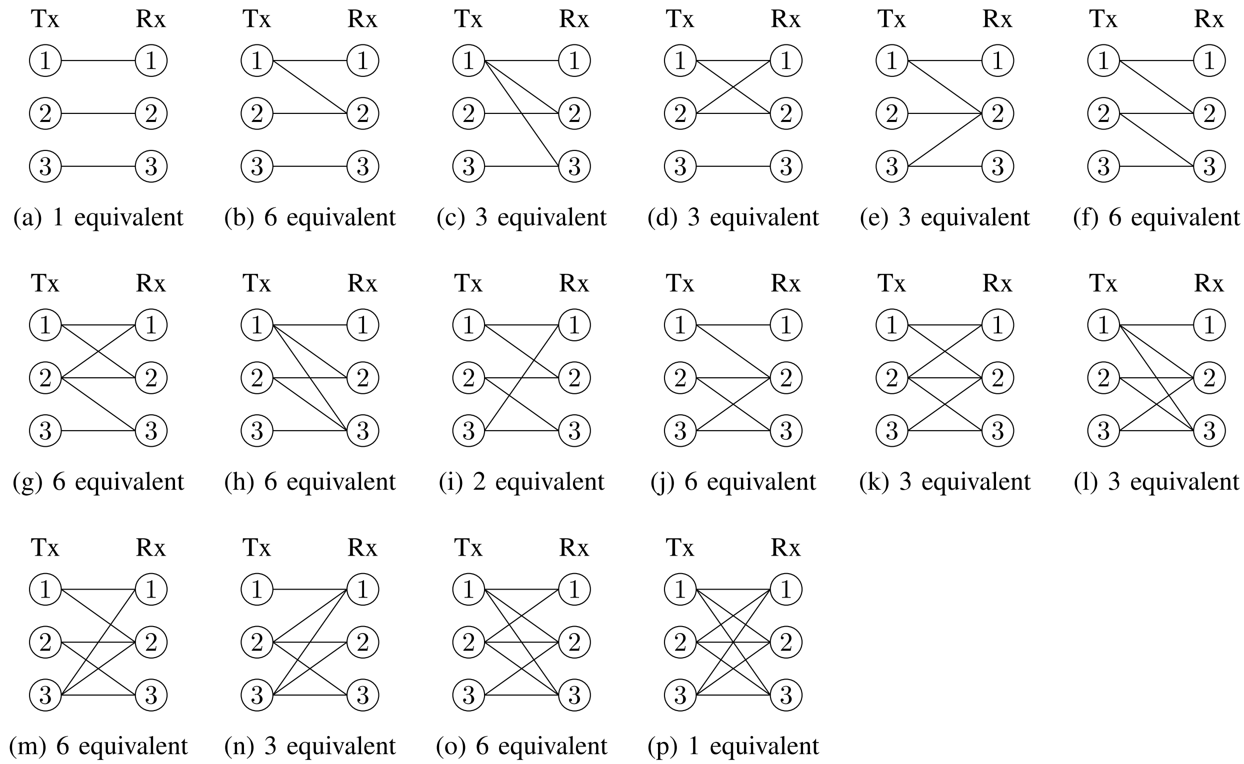

Fig. 3. All possible canonical network topologies in a three-user interference channel. The number below each topology is the number of equivalent topologies for that case.

be t sets, $A_i \subseteq \{1, 2, \cdots, K\}$ for all $1 \leq i \leq t$, such that each user is in at-least one A_i. We consider t time-slots, where in time-slot $1 \leq i \leq t$, users in set A_i are scheduled and they send data at rate equal to the capacity of the direct link to the destination. If a user is scheduled in more time-slot, it repeats the information across time-slots. We constrain the sets such that if this strategy is used, the data can be decoded at the receivers for all possible set of network states. The selection of optimal strategy is to minimize t. This optimal strategy is called CMIS Scheduling.

Theorem 5. *CMIS Scheduling is optimal for any 3-user interference channel, cyclic chain, one-to-many interference channel, many-to-one interference channel and Z-chain when all the transmitters have one hop information.*

MIG Scheduling is also optimal for many classes of networks with 2 hops information as depicted in the following theorem

Theorem 6. *MIG Scheduling is optimal for $h = 2$ when the network is of one of the following forms*

1) *Two user interference channel, $\alpha^*(2) = 1$,*
2) *Z-channel chain, $\alpha^*(2) = 2/3$ for $K \geq 2$,*
3) *d-to-many interference channel, $\alpha^*(2) = \frac{d}{2d-1}$ for $1 \leq d < K$ and $K \geq 2$,*
4) *many-to-one interference channel, $\alpha^*(2) = \frac{K-1}{2K-3}$ for $K \geq 2$,*
5) *fully-connected interference channel, $\alpha^*(2) = 1$.*

We will now describe the performance of MIG Scheduling for three-user interference channels when there is 2-hop information at the transmitters. Three user interference channel consists of sixteen possible networks upto symmetry which are described in Figure 3. The next theorem describes the cases when MIG Scheduling is optimal.

Theorem 7. *MIG Scheduling is optimal with 2-hops information when the network is any of the configurations (a), (b), (c), (d), (e), (f), (h), (i), (j), (m), (n), (p).*

This theorem shows that the MIG schedule is optimal for 12 out of 16 possible 3-user topologies. Next theorem computes the difference between MIG Scheduling and the optimal strategy [6] for topology (g).

Theorem 8. *If the network is of the form (g), MIG Scheduling achieves $\alpha(2) = 2/3$ while $\alpha^*(2) = 4/5$.*

The optimal strategy achieving $\alpha^*(2) = 4/5$ exploits the fact that the second transmitter knows the complete network topology with two hops of information. Thus, it is better for the second transmitter to transmit even when first and the third transmitter are transmitting at the direct channel capacity from then to their respective destination. Hence, better strategies as compared to MIG scheduling can be applied.

V. Conclusions

In this paper, we give a framework for optimality of distributed decisions. This framework gives the optimal normalized sum-capacity if the network is known upto a limited hops of information. We also characterize strategies that can be used, and prove their optimality in many cases of network topologies. The proofs of the results in this paper and its extension to Gaussian networks can be found in [6].

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