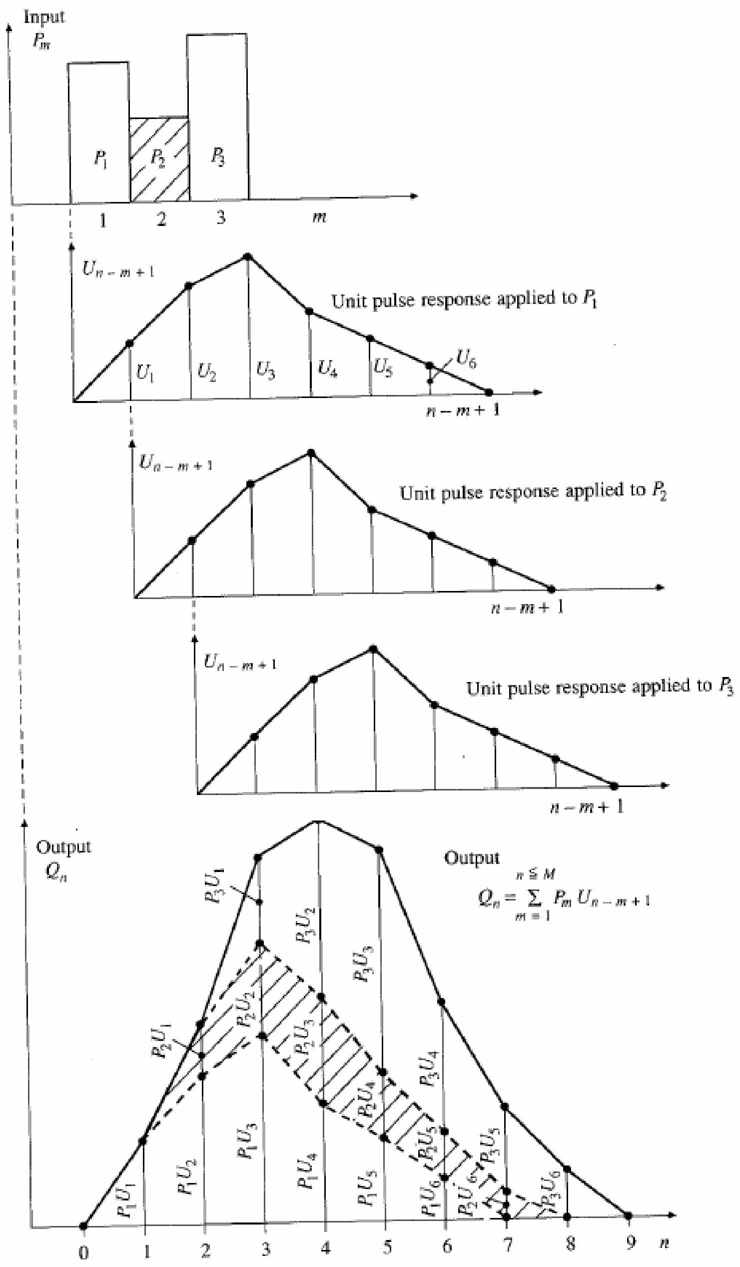
Unit Hydrograph – Convolution Equation

Application of Linear Systems approach



Linear System – Discrete Time

- Time intervals for excess rainfall $m = 1,2,\cdots,M$
- Time intervals for direct runoff $n = 1,2,\cdots,N$
- Excess rainfall during m^{th} time interval $P_m \; m = 1,2,\cdots,M$
- Streamflow at end of n^{th} time interval $Q_n \; n = 1,2,\cdots,N$

- Response of a linear system is the sum (convolution) of the responses to inputs that have happened in the past.

$$Q(t) = \int_0^t u(t-\tau)d\tau$$

Continuous time

$$Q_n = \sum_{m=1}^{n\leq M} P_m U_{n-m+1}$$

Discrete time

Expanding the Convolution Equation

$$Q_n = \sum_{m=1}^{n \le M} P_m U_{n-m+1}$$

$$\begin{aligned}
Q_1 &= P_1 U_1 \\
Q_2 &= P_2 U_1 + P_1 U_2 \\
Q_3 &= P_3 U_1 + P_2 U_2 + P_1 U_3 \\
&\dots \\
Q_M &= P_M U_1 + P_{M-1} U_2 + \dots + P_1 U_M \\
Q_{M+1} &= 0 + P_M U_2 + \dots + P_2 U_M + P_1 U_{M+1} \\
&\dots \\
Q_{N-1} &= 0 + 0 + \dots + 0 + 0 + \dots + P_M U_{N-M} + P_{M-1} U_{N-M+1} \\
Q_N &= 0 + 0 + \dots + 0 + 0 + \dots + 0 + P_M U_{N-M+1}
\end{aligned}$$

Deriving unit hydrograph from Convolution Equation

- Q_n and P_m are given, derive unit hydrograph (UH) (lets say M = 3 and N = 6)
- UH will have N-M+1 = 4 pulses
- Use the discrete convolution to write down the equations

$$Q_n = \sum_{m=1}^{n \le M} P_m U_{n-m+1}$$

$$Q_1 = P_1 U_{1-1-+1} = P_1 U_1$$

$$Q_2 = P_1 U_{2-1+1} + P_2 U_{2-2+1} = P_2 U_1 + P_1 U_2$$

$$Q_3 = P_1 U_{3-1+1} + P_2 U_{3-2+1} + P_3 U_{3-3+1} = P_3 U_1 + P_2 U_2 + P_1 U_3$$

$$Q_4 = P_1 U_{4-1+1} + P_2 U_{4-2+1} + P_3 U_{4-3+1} = P_3 U_2 + P_2 U_3 + P_1 U_4$$

Then solve for U_1, U_2, U_3. and U_4 as below

$$U_1 = \frac{Q_1}{P_1} \qquad U_2 = \frac{Q_2 - P_2 U_1}{P_1} \qquad U_3 = \frac{Q_3 - P_2 U_1 - P_2 U_2}{P_1}$$

$$U_4 = \frac{Q_4 - P_3 U_2 - P_2 U_3}{P_1}$$

Deconvolution