Extending Optimal Oblivious Reconfigurable Networks to all $N$

APOCS 2023

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How do we connect servers so they can communicate?
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How do we route messages along those connections?
Oblivious Reconfigurable Networks (ORNs)

• Set of $N$ nodes
• Edges reconfigure between each timestep according to a predefined schedule
• Route messages obliviously
  • Co-designing a connection schedule and routing protocol
Oblivious Reconfigurable Networks
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![Diagram](image-url)
Oblivious Reconfigurable Networks
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Oblivious Reconfigurable Networks

\[ \begin{array}{cccc}
\text{a,1} & \text{a,2} & \text{a,3} \\
\text{b,1} & \text{b,2} & \text{b,3} \\
\text{c,1} & \text{c,2} & \text{c,3} \\
\text{d,1} & \text{d,2} & \text{d,3} \\
\end{array} \]
Oblivious Reconfigurable Networks
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Route $a \rightarrow c$ starting at $t = 1$
Oblivious Reconfigurable Networks

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Oblivious Reconfigurable Networks

Route $a \rightarrow c$ starting at $t = 1$
Path has latency $L = 2$
Throughput

\[
D_t
\]
Throughput

Demand from $i \rightarrow j$ at timestep $t$
Throughput

• A matrix requests throughput \( r \)
Throughput

- A matrix requests throughput $r$
Throughput

- A matrix requests throughput $r$
- An ORN design guarantees throughput $r$ if it can route all matrices requesting throughput $r$ without overloading edges
The Problem

• Build an ORN design with:
  • High guaranteed throughput $r$
  • Low max latency $L$

• These objectives are in conflict with each other!
  • So looking for a tradeoff
Theorem: Let $0 < r \leq \frac{1}{2}$ be a constant, and $h = \left\lfloor \frac{1}{2r} \right\rfloor$, and

$$
\varepsilon = h + 1 - \frac{1}{2r} \in (0,1], \text{ and let } L^*(r, N) \text{ be the function}
$$

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L^*(r, N) = h(N^{1/(h+1)} + (\varepsilon N)^{1/h})
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Then for every ORN design on \(N\) nodes that guarantees throughput \(r\), the maximum latency is at least \(\Omega(L^*(r, N))\).

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Furthermore for infinitely many \(N\), there exists an ORN design on \(N\) nodes that guarantees throughput \(r\) and whose maximum latency is \(O(L^*(r, N))\).

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Valiant Load Balancing\textsuperscript{2}

- Given routing protocol $R$ for the uniform demand matrix $D_{\text{unif}}(2r)$
- To route throughput $r$ obliviously from $a \rightarrow b$, choose a random intermediate node $c$ and use $R$ to route from $a \rightarrow c$ then $c \rightarrow b$

\textsuperscript{2}Leslie G. Valiant. A scheme for fast parallel communication. *SIAM J Comput.* ‘82
Valiant Load Balancing

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The Elementary Basis Scheme (EBS)

\[ N = a \text{ perfect square} \]
The Elementary Basis Scheme (EBS)

Phase 1
groups
The Elementary Basis Scheme (EBS)

Phase 2 groups
(0,0)→(1,2)

- Choose intermediate (2,1)
(0,0) → (1,2)

• Choose intermediate (2,1)
Choose intermediate (2,1)
(0,0)→(1,2)
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Guarantees throughput $r = \frac{1}{4}$
(0,0) → (1,2)
- Choose intermediate (2,1)

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Max latency $L = 4\sqrt{N}$
(0,0) → (1,2)

- Choose intermediate (2,1)

Guarantees throughput $r = \frac{1}{4}$

Max latency $L = 4\sqrt{N} \leq O(L^*(r, N))$
When $N$ is Not a Perfect Square

- Inflate $N$ to the next largest perfect square $M$
- Denote $(M - N)$ nodes as “dummy nodes”
- Ignore flow on routing paths that would go through dummy nodes
- Show this doesn’t decrease throughput too much
General EBS

• $a \in [N] \rightarrow h$-tuples $\in [N^{1/h}]^h$
• Split period into $h$ phases, one for each index of the tuples
• Semi paths use $\leq 1$ hop per phase over next $h$ phases
  • Apply VLB to the semi-paths
• Guarantees throughput $\frac{1}{2h}$
• Max latency $2hN^{1/h} \leq O(L^*(\frac{1}{2h}, N))$
• Achieves most optimal throughput-latency tradeoff points
Choosing a Dummy Node Set

\[ \left\{ \left( i_1, i_2, \ldots, i_{h-1}, \sum_{j=1}^{h-1} i_j \right) : i_1, \ldots, i_{h-1} \in [M^{1/h}], \right\} \]

Exactly 1 node per phase group
Choosing a Dummy Node Set

\[ \mathcal{D} = \left\{ \left( i_1, i_2, \ldots, i_{h-1}, \ell + \sum_{j=1}^{h-1} i_j \right) : i_1, \ldots, i_{h-1} \in \left[ M^{1/h} \right], \ell \in [h] \right\} \]

- Exactly 1 node per phase group
- \( h \) different diagonals
The Vandermonde Basis Scheme (VBS)

• Defines phase connections using Vandermonde vectors
  • Allows greater flexibility in semi-path choice, allowing fine-tuning when EBS fails

• Treat nodes as vectors in an \((h + 1)\)-dimensional vector space over \(\mathbb{F}_q\) for \(q = N^{1/(h+1)}\)
  • So \(N\) must be a prime \((h + 1)\)-power

• Define “diagonal” set carefully to keep it well distributed across the Vandermonde phase groups

• Use a prime gap theorem\(^3\) to bound number of “diagonals” we need

Putting Everything Together

• Want: guarantee throughput $r$ for arbitrary number of nodes

• Then need to build a design which can guarantee $r' > r$ throughput without dummy nodes

• This design will achieve max latency $O\left( L^*(r', M) \right)$

• Then show that $O\left( L^*(r', M) \right) \leq O\left( L^*(r, N) \right)$

  - This is possible when $r$ is not an even integer

• Right derivative of $L^*$ is too steep at this point

• Open Q: can we fix this?
Future Directions & Open Problems

• Address problems that arise when you remove theoretical assumptions
  • No fractional flow → queueing and congestion control
  • Propagation delay
• Node failures
• If we know the workload when routing, can we do better?
Thank You!

Questions?